Hamiltonian decompositions of the wreath product of two hamiltonian decomposable directed graphs

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Hamiltonian decomposable

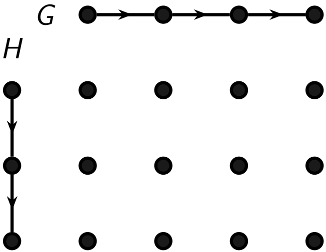
Definition

A graph (directed graph) is **hamiltonian decomposable** if it admits a decomposition into (directed) hamiltonian cycles.

Wreath product

Definition

The **wreath product** of G and H, denoted $G \wr H$, is a digraph on vertex set $V(G) \times V(H)$, where $((x, y), (u, v)) \in A(G \wr H)$ if and only if...



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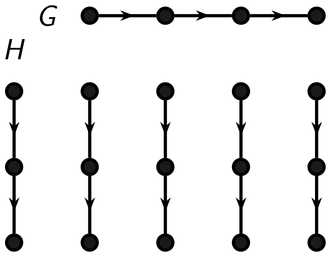

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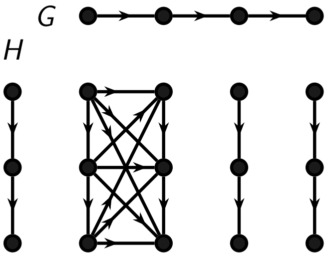

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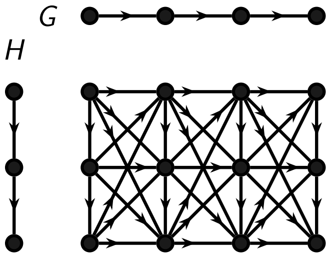

Main problem

Question: Given two hamiltonian decomposable (directed) graphs G and H, is $G \wr H$ also hamiltonian decomposable?

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If G and H are hamiltonian decomposable graphs, then $G \wr H$ is also hamiltonian decomposable.

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Theorem (Ng, 1998)

If G and H are hamiltonian decomposable digraphs, $|V(G)|$ is odd, and $|V(H)| > 2$, then $G \wr H$ is also hamiltonian decomposable.

Main question refined

Question: Given two hamiltonian decomposable digraphs graphs G and H, such that $|V(G)|$ is even, is $G \wr H$ also hamiltonian decomposable?

Reduction

Proposition (Ng, 1998)

Let G and H be hamiltonian decomposable directed graphs such that $|V(G)| = n$ and $|V(H)| = m$. If

1. $\vec{C}_n \wr H$ *is hamiltonian decomposable,*
2. *and* $\vec{C}_n \wr \overline{K}_m$ *are hamiltonian decomposable,*

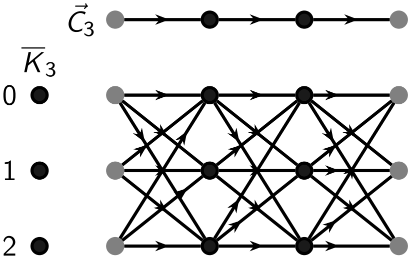
then $G \wr H$ *is hamiltonian decomposable.*

Note that $\vec{C}_n$ denotes the directed cycle on n vertices.

The directed graph $\vec{C}_n \wr \overline{K}_m$

Lemma (Ng, 1998)

If $m \geqslant 3$, then $\vec{C}_n \wr \overline{K}_m$ is hamiltonian decomposable.



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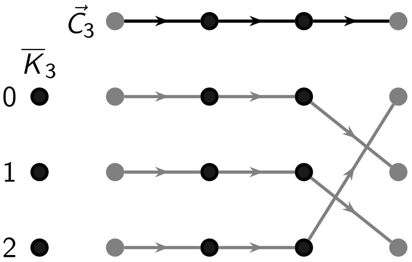

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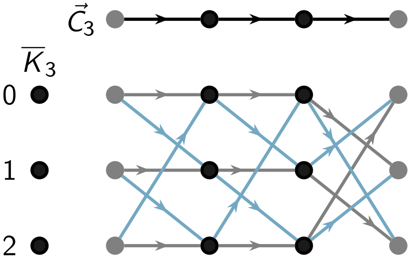

$$F_0 = (id, id, (0,1,2));$$
$$F_1 = ((0,1,2),(0,1,2),(0,2,1)).$$

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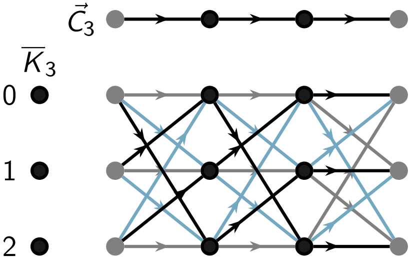

$$\begin{aligned} F_0 &= (id, id, (0,1,2)); \\ F_1 &= ((0,1,2),(0,1,2),(0,2,1)); \\ F_2 &= ((0,2,1),(0,2,1),id). \end{aligned}$$

2-factorization of $\vec{C}_n \wr \overline{K}_m$

Each 2-factorization of $\vec{C}_n \wr \overline{K}_m$ can be described as a set of m n-tuples of permutations from S_m:

$$\mathcal{F} = \left\{ \begin{array}{llll} (\mu_{(0,0)}, & \mu_{(0,1)}, & \ldots, & \mu_{(0,n-1)}); \\ (\mu_{(1,0)}, & \mu_{(1,1)}, & \ldots, & \mu_{(1,n-1)}); \\ & \vdots & & \\ (\mu_{(m-1,0)}, & \mu_{(m-1,1)}, & \ldots, & \mu_{(m-1,n-1)}). \end{array} \right\}$$

Decomposition families

Definition

Let $T = \{\mu_{(0,j)}, \mu_{(1,j)}, \ldots, \mu_{(m-1,j)}\}$ be a set of m permutations from the symmetric group S_m. The set T is a **decomposition family of order** m if $\mu_{(k_1,j)}\mu^{-1}_{(k_2,j)}$ is a derangement for all $\mu_{(k_1,j)} \neq \mu_{(k_2,j)}$.

Example:

$$\mathcal{F} = \left\{ \begin{array}{lll} (id, & id, & (0,1,2)) \\ ((0,1,2), & (0,1,2), & (0,2,1)) \\ ((0,2,1), & (0,2,1), & id) \end{array} \right\}$$

Hamiltonian n-tuple

Definition

Let $\mu_{(i,0)}, \mu_{(i,1)}, \ldots, \mu_{(i,n-1)} \in S_m$. The n-tuple $(\mu_{(i,0)}, \mu_{(i,1)}, \ldots, \mu_{(i,n-1)})$ is a **hamiltonian n-tuple** if

$$\tau_i = \mu_{(i,0)}\mu_{(i,1)} \cdots \mu_{(i,n-1)}$$

is a permutation on a single cycle.

Example:

$$F_0 = (id, id, (0, 1, 2)) \Rightarrow \tau_0 = (0, 1, 2).$$

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Example:

$$F_0 = (id, id, (0,1,2)) \Rightarrow \tau_0 = (0,1,2);$$
$$F_1 = ((0,1,2),(0,1,2),(0,2,1)) \Rightarrow \tau_1 = (0,1,2)(0,1,2)(0,2,1) = (0,1,2).$$

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$$F_2 = ((0,2,1),(0,2,1),id) \Rightarrow \tau_2 = (0,2,1)(0,2,1) = (0,1,2).$$

In summary

The digraph $\vec{C}_n \wr \overline{K}_m$ is hamiltonian decomposable if we have

$$\left.\begin{array}{llll}(\mu_{0,0}, & \mu_{0,1}, & \ldots, & \mu_{0,n-1});\\ (\mu_{1,0}, & \mu_{1,1}, & \ldots, & \mu_{1,n-1});\\ \vdots & \vdots & \vdots & \vdots\\ (\mu_{m-1,0}, & \mu_{m-1,1}, & \ldots, & \mu_{m-1,n-1}).\end{array}\right\} m \text{ hamiltonian } n\text{-tuples}$$

where $\{\mu_{(0,i)}, \mu_{(1,i)}, \ldots \mu_{(m-1,i)}\}$ is a decomposition family of order m for each $i \in \mathbb{Z}_n$.

Hamiltonian decomposition of $\vec{C}_n \wr H$

We will take a similar approach for the digraph $\vec{C}_n \wr H$:

$$\left.\begin{array}{llll} (\mu_{0,0}, & \mu_{0,1}, & \dots, & \mu_{0,n-1}); \\ (\mu_{1,0}, & \mu_{1,1}, & \dots, & \mu_{1,n-1}); \\ \vdots & \vdots & \vdots & \vdots \\ (\mu_{m-1,0}, & \mu_{m-1,1}, & \dots, & \mu_{m-1,n-1}). \end{array}\right\} m \ n\text{-tuples such that...}$$

where $\{\mu_{(0,i)}, \mu_{(1,i)}, \dots \mu_{(m-1,i)}\}$ is a decomposition family of order m for each $i \in \mathbb{Z}_n$.

Truncation of a permutation

Definition

Let $\mu \in S_m$ be such that $(m-1)^{\mu} \neq m-1$. The **truncation** of μ, denoted $\hat{\mu}$, is the permutation

$$\hat{\mu} = \mu\,(m-1, (m-1)^{\mu}).$$

Example: $\mu = (0, 1, 2, 3, 4, 5, 6, 7) \in S_8$.

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$$\sigma_i = \hat{\mu}_{(i,0)}\hat{\mu}_{(i,1)} \cdots \hat{\mu}_{(i,n-1)}$$

is a permutation with two cycles in its disjoint cycle notation.

Example: $((0,2),(0,2),(0,1,2))$, where $(0,2),(0,1,2) \in S_3$

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Example:$((0, 2), (0, 2), (0, 1, 2))$;

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General Approach

Let H be a digraph on m vertices that admits a decomposition into c directed hamiltonian cycles ($1 \leqslant c \leqslant m-2$). The digraph $\vec{C}_n \wr H$ is hamiltonian decomposable if we have:

$$\left.\begin{matrix} (\mu_{0,0}, & \mu_{0,1}, & \dots, & \mu_{0,n-1}); \\ (\mu_{1,0}, & \mu_{1,1}, & \dots, & \mu_{1,n-1}); \\ & \vdots & & \\ (\mu_{c-1,0}, & \mu_{c-1,1}, & \dots, & \mu_{c-1,n-1}); \end{matrix}\right\} c \text{ truncated hamiltonian } n\text{-tuples}$$

$$\left.\begin{matrix} (\mu_{c,0}, & \mu_{c,1}, & \dots, & \mu_{c,n-1}); \\ (\mu_{c+1,0}, & \mu_{c+1,1}, & \dots, & \mu_{c+1,n-1}); \\ & \vdots & & \\ (\mu_{m-1,0}, & \mu_{m-1,1}, & \dots, & \mu_{m-1,n-1}). \end{matrix}\right\} m-c \text{ hamiltonian } n\text{-tuples}$$

One more reduction step

Proposition

Let G and H be hamiltonian decomposable directed graphs such that $|V(G)| = n$ is even. If $\vec{C}_2 \wr H$ is hamiltonian decomposable then $\vec{C}_n \wr H$ is hamiltonian decomposable.

Summary: It suffices to show that $\vec{C}_2 \wr H$ is hamiltonian decomposable

Consequences

Let H be a digraph on m vertices that admits a decomposition into c directed hamiltonian cycles ($1 \leqslant c \leqslant m-2$). The digraph $\vec{C}_2 \wr H$ is hamiltonian decomposable if there exists m pairs of permutations such that:

$$\left.\begin{array}{l}(\mu_0, \tau_0);\\ (\mu_1, \tau_1);\\ \quad\vdots\\ (\mu_{c-1}, \tau_{c-1});\end{array}\right\} c \text{ truncated hamiltonian pairs}$$

$$\left.\begin{array}{l}(\mu_c, \tau_c);\\ (\mu_{c+1}, \tau_{c+1});\\ \quad\vdots\\ (\mu_{m-1}, \tau_{m-1}).\end{array}\right\} m-c \text{ hamiltonian pairs}$$

Solution for the case for $m = 13$ and $c = 2$

If H is a digraph on $m = 13$ vertices that admits a decomposition into 2 hamiltonian cycles, then we aim to construct a set of 13 pairs of permutations.

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Step 1: To construct two decomposition families.

The decomposition family $\mathcal{F}_{13}$

$$\mathcal{F}_{13} = \left\{ \begin{array}{l} \sigma_1 = (0,1,12,2,3,4,5,6,7,8,9,10,11); \\ \sigma_2 = (0,2,4,6,12,8,10)(1,3,5,7,9,11); \\ \sigma_3 = (0,12,3,6,9)(1,4,7,10)(2,5,8,11); \\ \sigma_4 = (0,4,8)(1,5,12,9)(2,6,10)(3,7,11); \\ \sigma_5 = (0,5,10,3,8,1,6,11,12,4,9,2,7); \\ \sigma_6 = (0,6)(1,7)(2,8)(3,9)(4,12,10)(5,11); \\ \sigma_7 = (0,7,2,9,4,11,6,1,8,3,10,12,5); \\ \sigma_8 = (0,8,4)(1,9,5)(2,10,6)(3,12,11,7); \\ \sigma_9 = (0,9,12,6,3)(1,10,7,4)(2,11,8,5); \\ \sigma_{10} = (0,10,8,6,4,2,12)(1,11,9,7,5,3); \\ \sigma_{11} = (0,11,10,9,8,12,7,6,5,4,3,2,1); \\ \sigma_{12} = (0,3,11,4,10,5,9,6,8,7,12,1,2); \\ \sigma_0 = id. \end{array} \right\}$$

Solution for the case for $m = 13$ and $c = 2$

If H is a digraph on $m = 13$ vertices that admits a decomposition into 2 hamiltonian cycles, then we aim to construct a set of 13 pairs of permutations.

Step 1: To construct two decomposition families.

Step 2: We construct a set of 13 pairs of permutations using elements of $\mathcal{F}_{13} \times \mathcal{F}_{13}$.

Hamiltonian array of $\mathcal{F}_{13} \times \mathcal{F}_{13}$

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_0
σ_1													
σ_2													
σ_3													
σ_4													
σ_5													
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	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_0
σ_1	■								■				■
σ_2								■					
σ_3							■				■		
σ_4						■				■			
σ_5					■				■				■
σ_6				■				■					
σ_7			■				■						■
σ_8		■				■							
σ_9	■				■								
σ_{10}				■									
σ_{11}			■								■		■
σ_{12}												■	■
σ_0	■				■		■				■	■	

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	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_0
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	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_0
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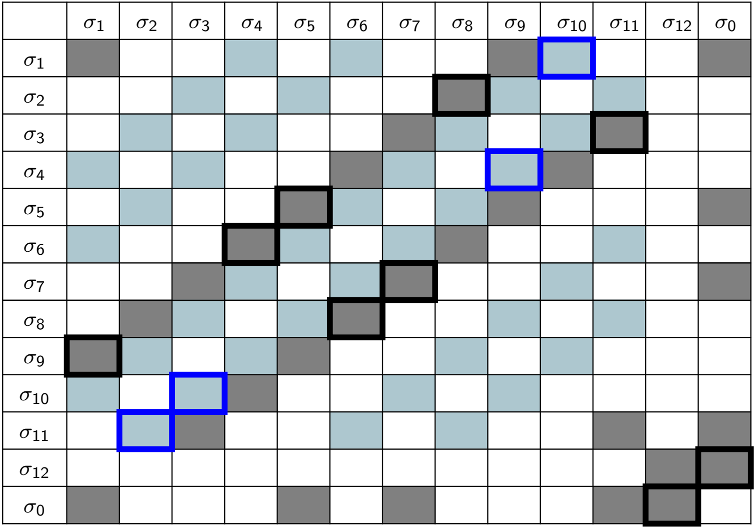
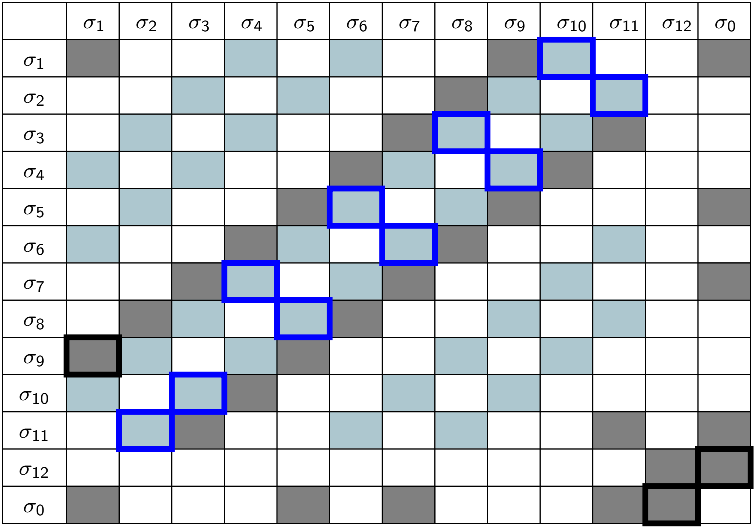
Solution for $m = 13$ and $c = 4$



Solution for $m = 13$ and $c = 10$



Summary of results

Theorem

Let G and H be hamiltonian decomposable directed graphs such that $|V(H)| > 3$ and $|V(G)|$ is even. Then $G \wr H$ is hamiltonian decomposable except possibly when

1. *G is a directed cycle,*
2. *$|V(H)|$ is even,* **and**
3. *H admits a decomposition into an odd number of directed hamiltonian cycles.*

Proposition

If n is even, then $\vec{C}_n \wr \vec{C}_2$ and $\vec{C}_n \wr \vec{C}_3$ are not hamiltonian decomposable.

Thanks!