The game of Cops and Attacking Robber

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PIMS

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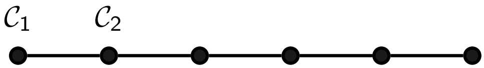

The game of Cops and Robber played on graphs

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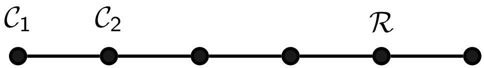
1. A set of $k > 0$ cops: $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$;
2. A single robber, $\mathcal{R}$.



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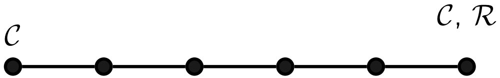
The game of Cops and Robber played on graphs

1. The cops first place themselves on vertices of the graph;
2. The robber then places themselves on a vertex;
3. Both sides play with perfect information;
4. During their turn, each cop makes a move by sliding across one edge or passing;
5. Likewise for the robber;
6. The cops win if one of them occupies the same vertex as the robber.



Some key terminology

Definition 1

A graph G is **reflexive** *if there exists a loop at every vertex of G.*

All graphs in the original game of Cops and Robber are assumed to be reflexive. This means that cops and robber alike can chose to pass on their turn.

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Removing loops from the vertices gives rise to the game of Active Cops and Robber where players are not allowed to pass.

Some key terminology

Definition 2

The **open neighbourhood** of vertex $v \in V(G)$, is defined as

$$N(v) = \{w \mid w \sim v, w \neq v\}.$$

The **closed neighbourhood** of vertex $v \in V(G)$, is defined as

$$N[v] = \{w \mid w \sim v\}.$$

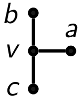

Figure: Open neighbourhood of v is $\{a, b, c\}$. Closed neighbourhood of v is $\{v, a, b, c\}$.

Copwin graphs

Definition 3

A graph G is **copwin** *if a single cop has a winning strategy on G.*

Figure: A path is a copwin graph.

Copwin graphs

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A graph G is **copwin** if a single cop has a winning strategy on G.

Copwin graphs are characterized by Nowakowski & Winkler (1983) and Quilliot (1983).

Definition 4

A vertex v of G is a **corner** if there exists a vertex $w \neq v$ such that $N[v] \subseteq N(w)$.

Characterizing copwin graphs

Lemma 1

A graph is copwin only if it contains a corner.

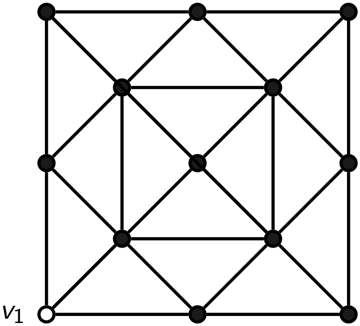

Figure: A corner drawn in white.

Characterizing copwin graphs

Lemma 1

Let v be a corner of graph G such that $|V(G)| > 1$; then G is copwin if and only if $G - \{v\}$ is copwin.

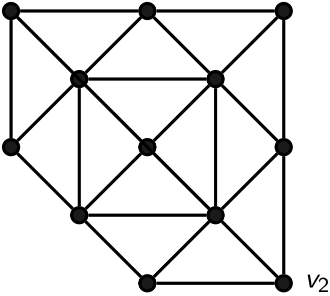

Figure: The graph $G - \{v_1\}$

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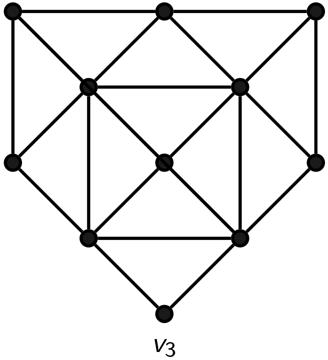

Figure: The graph $G - \{v_1, v_2\}$

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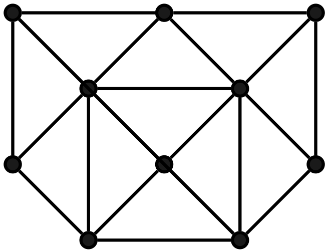

Figure: The graph $G - \{v_1, v_2, v_3\}$

We repeat this process until we are left with one vertex.

Characterizing copwin graphs

Definition 5

A graph G on n vertices is **dismantlable** if there exists an ordering of its vertices $v_1, v_2, \ldots, v_n$ such that v_i is a corner in $G_i = G - \{v_1, v_2, \ldots, v_{i-1}\}$. This ordering is known as a **domination ordering**.

Characterizing copwin graphs

Theorem 1 (Nowakowski and Winkler, 1983; Quilliot 1978)

A finite graph is copwin if and only if it is dismantlable.

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Clarke and Nowakowski (2001) defined an explicit copwin strategy as a function of the domination ordering.

Clarke et. al (2014) devised an algorithm that takes time $O(n^3)$ to determine if a graph is dismantlable.

Playing with multiple cops

Definition 6

In general, G is k-**copwin** if k cops have a winning strategy on G. The **copnumber** of a graph is defined as follows:

$$c(G) = \min\{k \mid \text{G is } k\text{-copwin}\}.$$

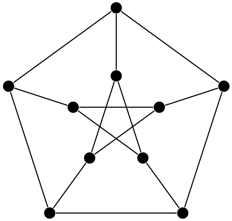


Figure: The Petersen graph has copnumber 3.

The Petersen graph

Example: The Petersen graph is a vertex-transitive graph.

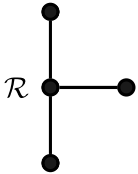
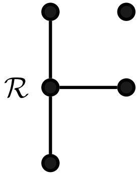


Figure: Open neighbourhood of the robber's position.

The Petersen graph

Example: The Petersen graph has girth five.



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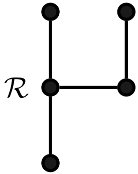

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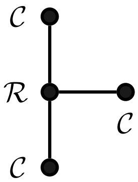

Figure: There must be a cop for every neighbour of a vertex.

Lower bounds

Proposition 1 (Aigner and Fromme (1984))

If G is a graph of girth at least five with minimum degree δ, then $c(G) \geqslant \delta$.

Proposition 2 (Bonato and Burgess (2010))

Let $t \geqslant 1$ be an integer. If G, with minimum degree δ, is $K_{2,t+1}$-free, then

$$c(G) \geqslant \frac{\delta}{2t}.$$

Meyniel's Conjecture

Conjecture 1 (Meyniel)

Let $c(n) = \max\{c(G) \mid G \textit{ is a connected graph on } n \textit{ vertices}\}$. Then $c(n) \in O(\sqrt{n})$.

- Best bound as of February 2026: $c(n) \in O(\frac{n}{2^{(1o(1))\sqrt{\log(n)}}})$ (Lu and Peng (2011) and Scott and Sudakov (2011)).
- Conjecture is known to hold for: Abelian Cayley graphs, graphs of diameter 2, random graphs, etc...
- If this bound holds, it is tight as there are several graphs whose copnumber lies in $O(\sqrt{n})$ (eg: incidence graphs of certain designs).

Some upper-bounds

Folklore: Let $\gamma(G)$ be the domination number of a graph:

$$c(G) \leqslant \gamma(G).$$

Theorem 2 (Joret et. al (2010))

Let G be a graph with tree-width $tw(G)$. Then $c(G) \leqslant \frac{tw(G)}{2} + 1$.

Theorem 3 (Bowler et. al (2021))

Let G be a graph with genus $g(G)$. Then $c(G) \leqslant \lfloor \frac{4g(G)}{3} \rfloor + \frac{10}{3}$.

Conjecture: (Shröder, 2001) $c(G) \leqslant g(G) + 3$.

k-copwin graphs

- Clarke and McGillivray (2012) characterized k-copwin graphs. They devised an $O(|V(G)|^2 k + 2)$-algorithm that can decide if a graph G is k-copwin.
- Deciding if a graph is k-copwin is EXPTIME-complete (Kinnersley, 2015).
- Petr et. al (2022) devised an algorithm that determines if a graph is k-copwin that runs in $O(k|V(G)|^{k+2})$.

Making life difficult for the cops

There exist numerous variations that impose some constraint on the cops:

- Restricting the cops' visibility.
- Speeding up the robber.
- Giving the robber the ability to defend themself.

The game of Cops and Attacking Robber

Introduced by Bonato et. al (2013).

We will follow the same rules as the original game of Cops and Robber with one modification.

The attacking robber: If the robber is adjacent to a cop at the beginning of his turn, he can **attack** this cop by moving on that cop's vertex and eliminate this cop from play.

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Definition 7

The **cc-number** of G, denoted $cc(G)$, is the minimum number of cops for which there exists a winning strategy in the game of Cops and Attacking Robber.

Question: How does the cc-number of G relate to $c(G)$?

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Question: How does the cc-number of G relate to $c(G)$?

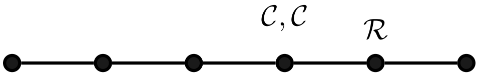
Simple examples such as trees, cycles, and graphs of small order suggest that $cc(G) \leqslant c(G) + 1$.

Obvious bounds

For all graphs G, we have:

$$c(G) \leqslant cc(G) \leqslant \min\{\gamma(G), 2c(G)\}.$$

With $2c(G)$ cops, the cops buddy up to protect themselves- making the attacking variant equivalent to the original game.



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Proposition 3 (Bonato et. al 2013)

A graph G is attacking copwin if and only if $\gamma(G) = 1$.

Results

Proposition 4 (Bonato et. al 2013)

If G is a connected outerplannar graph, then $cc(G) \leqslant 3$.

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Theorem 4 (Clow et. al 2025)

If G is a connected outerplannar graph such that $\gamma(G) > 1$, then $cc(G) = 2$ if and only if there exists an outerplanar embedding of G with at most one internal face of G of length $k \in \{5, 6\}$ and no internal faces of length greater than 6.

Further results

Theorem 5 (Clow et. al 2025)

If G is a bipartite connected planar graph, then $cc(G) \leqslant 4$. There exists a bipartite connected planar graph with cc-number 4.

Theorem 6

For every connected bipartite graph G, we have that $cc(G) \leqslant c(G) + 2$.

Question: Can we find a connected bipartite graph with

$$cc(G) = c(G) + 2?$$

Still open!

When is the $cc(G) = 2c(G)$?

Theorem 7 (Bonato et. al (2013))

There exists a graph for which $c(G) = 2$ and $cc(G) = 4$.

Proof:

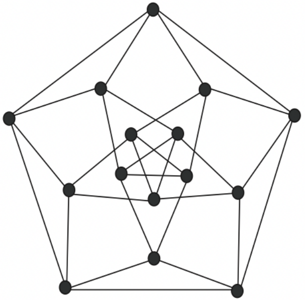

Figure: Line graph of Petersen graph

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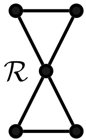

Figure: The closed neighbourhood of a vertex.

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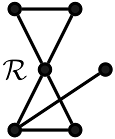

Figure: Two vertices in the open neighbourhood of a vertex v share no other neighbours.

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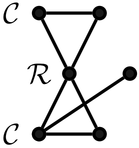

Figure: Two cops are needed in the original game (and will suffice).

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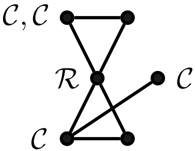


Figure: Each cop will require a back up with an attacking robber.

Therefore, we have that $cc(G) \geqslant 4$. By the trivial bound, $cc(G) \leqslant 4$, and we are done.

Generalization?

Question: (Bonato et. al, 2013) Does there exists other graphs for which $c(G) = k$ and $cc(G) = 2k$?

- For a positive integer k, construct a k-uniform hypergraph.

Lemma 1 (Bonato et. al (2013))

Let H be a k-uniform hypergraph with minimum degree at least 3 and girth at least 5. If the line graph of H, $L(H)$, has domination number at least $2k$, then $cc(L(H)) \geqslant 2k$.

Conjectures

Conjecture 2 (Clow et. al (2025))

For every $k \in \mathbb{N}$, there exists a connected graph such that $cc(G) - c(G) \geqslant k$.

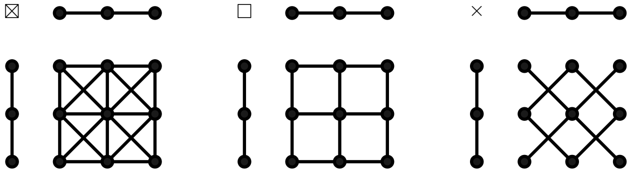
Conjecture 3 (Clow et. al (2025))

For every $k \in \mathbb{N}$, there exists a connected graph such that $c(G) = k$ and $cc(G) = 2k$.

Clow et. al (2025) construct a graph on 58 vertices with copnumber 3 and *cc*-number 6. They also provide 18 graphs for which $cc(G) - c(G) \geqslant 2$.

Graph products

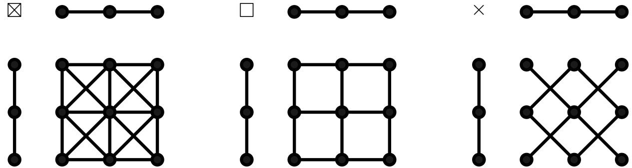
Graph products are binary operation between two graphs.



(a) Strong Product (⊠). (b) Cartesian product (□). (c) Categorical product ($\times$).

Figure: Illustration of $P_3 \otimes P_3$.

Graph products



(a) Strong Product ($\boxtimes$).

(b) Cartesian product ($\square$).

(c) Categorical product ($\times$).

Figure: Illustration of $P_3 \otimes P_3$.

Definition 8

Let $n, d \in \mathbb{N}$. The Hamming graph, $H(n, d)$, is the graph defined as

$$H(n, d) = \underbrace{K_n \square K_n \square \cdots \square K_n}_{d \text{ copies}}.$$

Cops and robber on graph products

Theorem 8 (Tošić (1987))

Let G and H be two non-trivial graphs. Then

$$c(G \Box H) \leqslant c(G) + c(H).$$

Theorem 9 (Neufeld and Nowakowski (1997))

Let $n, d \in \mathbb{N}$ such that $n \geqslant 3$. Then $c(H(n,d)) = d$.

Theorem 10 (Neufeld and Nowakowski (1997), Sullivan et. al (2010))

Let G and H be two non-trivial graphs. Then

$$c(G \boxtimes H) = c(G) + c(H) - 1.$$

Starting position

Unlike the original game of Cops and Robber, with an Attacking robber, the cops initial starting positions dictates whether they can play a winning strategy.

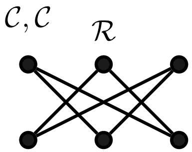
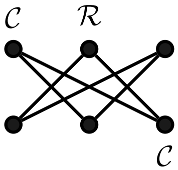


Figure: A poor choice of initial position for the cops on $K_3 \times K_2$.

Starting position

To win, the two cops must sit on a dominating set of size 2 of $K_3 \times K_2$.



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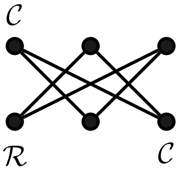

Figure: The cops are not on a dominating set of $K_3 \times K_2$.

Starting position

All dominating sets of size 2 of $K_3 \times K_2$ are independent sets.

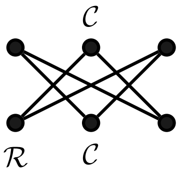

Figure: The cops move to a dominating set of $K_3 \times K_2$.

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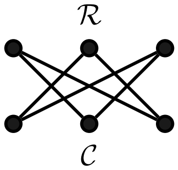

Figure: The robber eliminates a cop on $K_3 \times K_2$.

A similar observation holds for $K_2 \square K_2 \square K_2$.

Spreading strategy

Definition 9

The **spreading attacking copnumber** of G, denoted $cc_{\text{spread}}(G)$ is the smallest $k \in \mathbb{N}$, for which k cops can capture the robber in Cops and Attacking Robber with the additional constraint that all cops must start on the same vertex.

We have two graphs for which $cc(G) \neq cc_{\text{spread}}(G)$: $K_3 \times K_2$ and $K_2 \square K_2 \square K_2$.

Capturing an attacking robber on products

Proposition 5 (Haidar (2012))

Let G and H be two graphs. Then

$$cc(G \boxtimes H) \leqslant cc(G) + cc(H) - 1.$$

Proposition 6 (Brendle, Clow, LM, Ueckerdt, 2026+)

Let G and H be two graphs. Then

$$cc(G \boxtimes H) \leqslant cc_{spread}(G) + \mathbf{c}(\mathbf{H}) - 1.$$

The above bound is a significant improvement if $c(H) < cc(H)$.

Capturing an attacking robber on products

Question: Does $cc(G \square H) \leqslant cc(G) + cc(H)$ for all graphs G and H on more than two vertices?

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A first case: Let $n \geqslant 3$, then $cc(K_n \square K_n) = 3$.

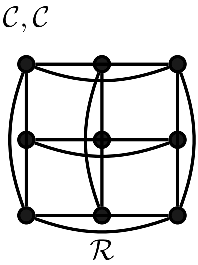

Figure: The graph $K_3 \square K_3$.

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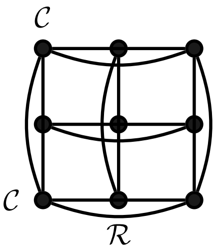

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A first case: Let $n \geqslant 3$, then $cc(K_n \square K_n) = 3$.

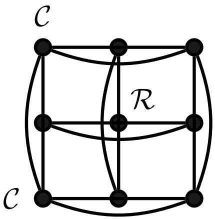


Figure: The graph $K_3 \square K_3$.

Capturing an attacking robber on products

Question: Does $cc(G \square H) \leqslant cc(G) + cc(H)$ for all graphs G and H on more than two vertices?

A first case: Let $n \geqslant 3$, then $cc(K_n \square K_n) = 3$.

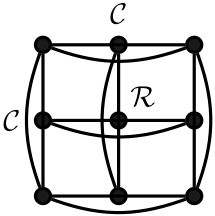


Figure: The graph $K_3 \square K_3$.

Capturing an attacking robber on products

Question: Does $cc(G\Box H) \leqslant cc(G) + cc(H)$ for all graphs G and H on more than two vertices?

A first case: Let $n \geqslant 3$, then $cc(K_n\Box K_n) = 3$.

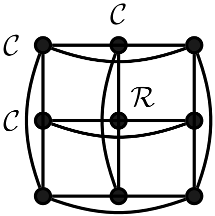


Figure: The graph $K_3\Box K_3$.

Therefore

$$cc(G\Box H) = 3 > cc(K_3) + cc(K_3).$$

Bounds on graph products

Theorem 11 (Brendle, Clow, LM, Ueckerdt, 2026+)

Let G and H be two graphs. Then

$$cc(G \square H) \leqslant cc(G) + cc(H) + 1.$$

Theorem 12 (Brendle, Clow, LM, Ueckerdt, 2026+)

Let G and H be two graphs such that $cc(G), cc(H) > 1$. Then

$$cc(G \square H) \leqslant cc_{spread}(G) + cc_{spread}(H).$$

Cartesian products

Theorem 13 (Brendle, Clow, LM, Ueckerdt, 2026+)

If $n, d \in \mathbb{N}$ such that $n \geqslant 3$, then

$$cc(H(n,d)) = \lceil \frac{d}{2} \rceil + d.$$

Therefore, we have that $cc(H(n,d)) - c(H(n,d)) = \lceil \frac{d}{2} \rceil$ for all $d \in \mathbb{N}$.

Cartesian products

Theorem 13 (Brendle, Clow, LM, Ueckerdt, 2026+)

If $n, d \in \mathbb{N}$ such that $n \geqslant 3$, then

$$cc(H(n,d)) = \lceil \frac{d}{2} \rceil + d.$$

Theorem 14 (Brendle, Clow, LM, Ueckerdt, 2026+)

If L_P be the line graph of the Petersen graph and let G be the n^{th} Cartesian product of L_P, then $cc(G) \geqslant \lceil \frac{n}{2} \rceil + 2n$.

Therefore, we have that $cc(G) - c(G) \geqslant \frac{n}{2}$ for $n \in \mathbb{N}$.

cc-number of Hamming graph

Theorem 15 (Brendle, Clow, LM, Ueckerdt, 2026+)

If $n, d \in \mathbb{N}$ such that $n \geqslant 3$, then

$$cc(H(n,d)) = \lceil \frac{d}{2} \rceil + d.$$

Proof:

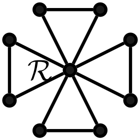


Figure: Open neighbourhood of vertex in $H(3,4)$.

cc-number of Hamming graph

Theorem 15 (Brendle, Clow, LM, Ueckerdt, 2026+)

If $n, d \in \mathbb{N}$ such that $n \geqslant 3$, then

$$cc(H(n,d)) = \lceil \frac{d}{2} \rceil + d.$$

Proof:

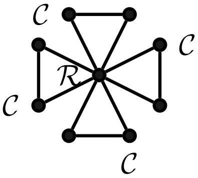


Figure: Four cops are needed in the original game.

cc-number of Hamming graph

Theorem 15 (Brendle, Clow, LM, Ueckerdt, 2026+)

If $n, d \in \mathbb{N}$ such that $n \geqslant 3$, then

$$cc(H(n,d)) = \lceil \frac{d}{2} \rceil + d.$$

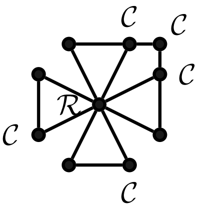


Figure: One cop can backup at most two cops adjacent to the robber.

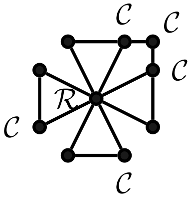
cc-number of Hamming graph

Theorem 15 (Brendle, Clow, LM, Ueckerdt, 2026+)

If $n, d \in \mathbb{N}$ such that $n \geqslant 3$, then

$$cc(H(n,d)) = \lceil \frac{d}{2} \rceil + d.$$

We will require at least $\lceil \frac{d}{2} \rceil$ additional cops.



cc-number of Hamming graph

Theorem 15 (Brendle, Clow, LM, Ueckerdt, 2026+)

If $n, d \in \mathbb{N}$ such that $n \geqslant 3$, then

$$cc(H(n,d)) = \lceil \frac{d}{2} \rceil + d.$$

Upper bound is done by induction on d and an application of the two bounds:

$$cc(G \Box H) \leqslant cc(G) + cc(H) + 1$$

$$cc(G \Box H) \leqslant cc_{spread}(G) + cc_{spread}(H).$$

Another example

Theorem 16 (Brendle, Clow, LM, Ueckerdt, 2026+)

For every $k \in \mathbb{N}$, there exists a graph G such that $c(G) = k$ and $cc(G) = 2k$.

Main result

Theorem 17 (Brendle, Clow, LM, Ueckerdt, 2026+)

For all $k \in \mathbb{N}$, there exists a graph G such that $cc(G) - c(G) \geqslant k$.

With the Hamming graph, we have a graph G such that $cc(G) = \frac{3}{2}c(G)$.

Theorem 18 (Brendle, Clow, LM, Ueckerdt, 2026+)

For all $k \in \mathbb{N}$, there exists a graph G such that $c(G) = k$ and $cc(G) = 2c(G)$.

Some open problems

Conjecture 4 (Clow et. al (2025))

If G is a graph such that $cc(G) - c(G) \geqslant 2$, then $|V(G)| \geqslant 15$.

Conjecture 5 (Clow et. al (2025))

If G is a graph such that $cc(G) - c(G) \geqslant 3$, then $|V(G)| \geqslant 58$.

Problem 1

Given $k \in \mathbb{N}$, what is the minimum order of G if $cc(G) - c(G) \geqslant k$?

Difficulties encountered on this projects highlight the need for a better understanding of the copnumber of graphs.

Thank you!

Prairie Discrete Mathematics Workshop:

- **Where:** University of Regina;
- **When:** May 7th and 8th, 2026;
- **Keynote speakers**: Jane Breen (Ontario Tech) and Melissa Huggan (Vancouver Island University)

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