Hamiltonian decompositions of the wreath product of directed cycles and complete symmetric digraphs

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PIMS

A puzzle

The setting: Consider a conference with 11 participants. To facilitate networking, the organizing committee decides to host 5 banquets. The banquet hall has one large round table.

The problem: The organizing committee needs a set of 5 seating arrangements (one for each banquet) such that each participant is seated to the right of every other participants exactly once.

Is this possible?

The wonderful Walecki Construction

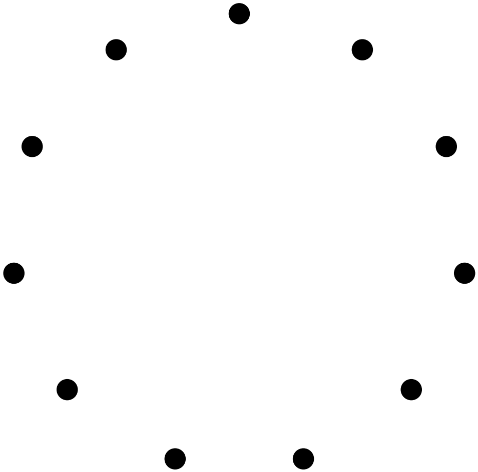


Figure: The 11 participants (one for each vertex).

The wonderful Walecki Construction

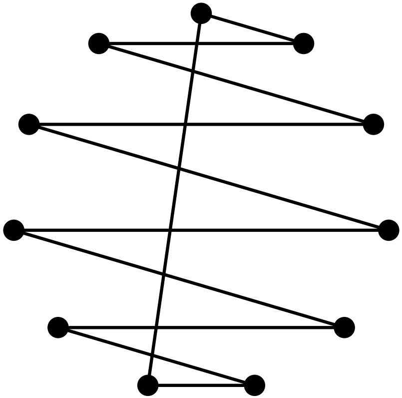


Figure: One seating arrangement with one table of length 11.

The wonderful Walecki Construction

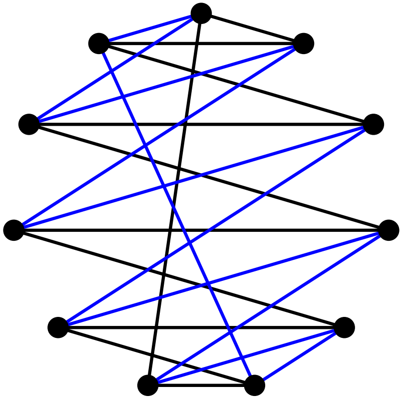


Figure: One seating arrangement with one table of length 11.

A more general puzzle

The setting: Consider a conference with n participants, with n being odd. To facilitate networking, the organizing committee decides to host $\frac{n-1}{2}$ banquets. The banquet hall has one large round table.

The problem: The organizing committee needs a set of $\frac{n-1}{2}$ seating arrangements (one for each banquet) such that each participant is seated to the right of every other participants exactly once.

Is this possible?

Hamiltonian decomposable

Definition

A graph (directed graph) is **hamiltonian decomposable** if it admits a decomposition into (directed) hamiltonian cycles.

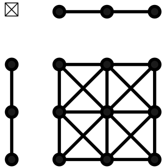
Question: If n is odd, is K_n (the complete graph) hamiltonian decomposable?

Hamiltonian decomposable graphs

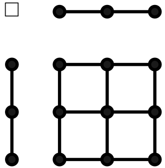
The study of hamiltonian decompositions of graphs has a rich history:

1. K_n when n is odd, and $K_n - I$ when n is even (Walecki, 1892);
2. the complete multipartite graph $K_{n[m]}$ when $n(m-1)$ is even (Laskar and Auerbach, 1976);
3. every hypercube is hamiltonian decomposable (Alspach, Bermond, and Sotteau, 1990) ;
4. Paley graphs (Alspach, Bryant, Dyer, 2012);
5. etc...

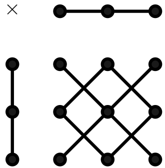
Graph Products



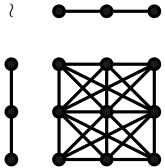
(a) Strong Product (⊠).



(b) Cartesian product (□).



(c) Categorical product (×).



(d) Wreath product (≀).

Decomposing graph products

In 1978, J.-C. Bermond proposed several interesting conjectures regarding hamiltonian decompositions of products of graphs. These gave rise to the following results:

$\boxtimes$ The strong product of two hamiltonian graphs is hamiltonian decomposable (Zhou 1989, Fan and Liu 1998);

$\wr$ The wreath product of two hamiltonian graphs is hamiltonian decomposable (Baranyai and Szás, 1981);

$\square$ The Cartesian product of two hamiltonian graphs is hamiltonian decomposable **under certain conditions** (Stong, 1991).

Hamiltonian decomposable directed graphs

We can also study directed hamiltonian decompositions of digraphs:

1. the complete symmetric digraph K_n^* is hamiltonian decomposable for all n (Tilson, 1980);
2. the complete symmetric multipartite graph $K_{n[m]}$ (Ng, 1997);
3. the symmetric directed hypercube is hamiltonian decomposable (Stong, 2006);
4. the Cartesian product of two directed cycles is hamiltonian decomposable if and only if very specific conditions are satisfied (Keating, 1978).

Decomposing products of directed graphs

For products of digraphs, less is known on their hamiltonian decompositions:

$\boxtimes$ the strong product of two hamiltonian digraph: ?;

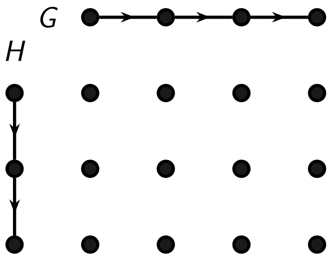
$\wr$ the wreath product of two hamiltonian digraphs is hamiltonian decomposable: almost all cases solved (Ng 1998, L-M 2025+) ;

$\square$ the Cartesian product of two hamiltonian digraphs: ?.

Wreath product

Definition

The **wreath product** of G and H, denoted $G \wr H$, is a digraph on vertex set $V(G) \times V(H)$, where $((x, y), (u, v)) \in A(G \wr H)$ if and only if...



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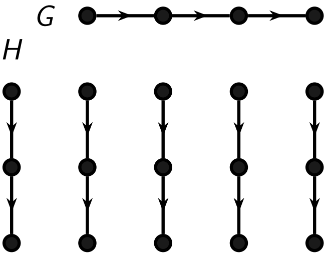

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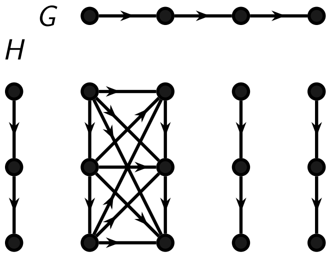

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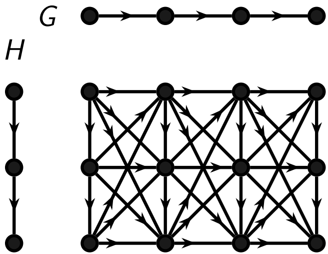

Main problem

Question: Given two hamiltonian decomposable (directed) graphs G and H, is $G \wr H$ also hamiltonian decomposable?

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Theorem (Baranyai and Szás, 1981)

If G and H are hamiltonian decomposable graphs, then $G \wr H$ is also hamiltonian decomposable.

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Question: Given two hamiltonian decomposable (directed) graphs G and H, is $G \wr H$ also hamiltonian decomposable?

Theorem (Baranyai and Szás, 1981)

If G and H are hamiltonian decomposable graphs, then $G \wr H$ is also hamiltonian decomposable.

Theorem (Ng, 1998)

If G and H are hamiltonian decomposable digraphs, $|V(G)|$ is odd, and $|V(H)| > 2$, then $G \wr H$ is also hamiltonian decomposable.

Main question refined

Question: Given two hamiltonian decomposable digraphs graphs G and H, such that $|V(G)|$ is even, is $G \wr H$ also hamiltonian decomposable?

Summary of results

Theorem (L-M (2025+))

Let G and H be hamiltonian decomposable directed graphs such that $|V(H)| > 3$ and $|V(G)|$ is even. Then $G \wr H$ is hamiltonian decomposable except possibly when

1. *G is a directed cycle,*
2. *$|V(H)|$ is even,* **and**
3. *H admits a decomposition into an odd number of directed hamiltonian cycles.*

Special cases

Theorem

Let G be a hamiltonian decomposable graph of even order and let $m \geqslant 4$. The digraph $G \wr \vec{C}_m$ is hamiltonian decomposable.

Theorem

The digraphs $\vec{C}_n \wr \vec{C}_3$ and $\vec{C}_n \wr \vec{C}_2$ are not hamiltonian decomposable when n is even.

Special cases

Theorem

Let G be a hamiltonian decomposable graph of even order and let $m \geqslant 3$. The digraph $G \wr K_m^*$ is hamiltonian decomposable.

Reduction

Proposition (Ng, 1998)

Let G and H be hamiltonian decomposable directed graphs such that $|V(G)| = n$ and $|V(H)| = m$. If

1. $\vec{C}_n \wr H$ *is hamiltonian decomposable,*
2. *and* $\vec{C}_n \wr \overline{K}_m$ *are hamiltonian decomposable,*

then $G \wr H$ is hamiltonian decomposable.

Reduction

Lemma (Ng, 1998)

The digraph $\vec{C}_n \wr \overline{K}_m$ is hamiltonian decomposable for all $n, m \geqslant 2$.

Conclusion: If we want to show that $G \wr \vec{C}_m$ and $G \wr K_m^*$ are hamiltonian decomposable, it suffices to show that $\vec{C}_n \wr \vec{C}_m$ and $\vec{C}_n \wr K_m^*$ are hamiltonian decomposable.

Decomposition of $\vec{C}_n \wr \vec{C}_m$

Proposition (L-M (2025))

Let n be even. The digraph $\vec{C}_n \wr \vec{C}_m$ is hamiltonian decomposable if and only if $m \geqslant 4$.

The digraphs $\vec{C}_n \wr \vec{C}_2$ and $\vec{C}_n \wr \vec{C}_3$ are not hamiltonian decomposable.

Decomposition of $\vec{C}_n \wr \vec{C}_3$

Proposition (L-M (2025))

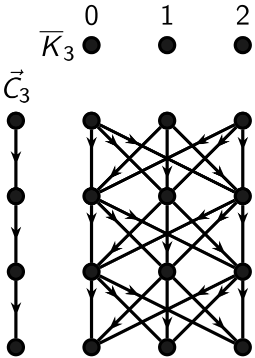
Let n be even. The digraph $\vec{C}_n \wr \vec{C}_3$ is not hamiltonian decomposable.

To describe the main idea behind our proof, we will look at the construction of a decomposition of $\vec{C}_n \wr \overline{K}_m$ into directed hamiltonian cycles.

The directed graph $\vec{C}_n \wr \overline{K}_m$

Lemma (Ng, 1998)

If $m \geqslant 3$, then $\vec{C}_n \wr \overline{K}_m$ is hamiltonian decomposable.



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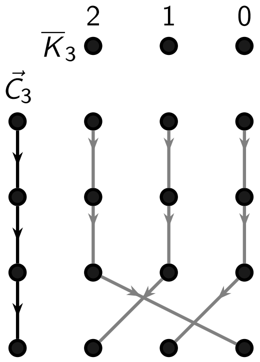

$F_0 = (id, id, (0, 1, 2))$

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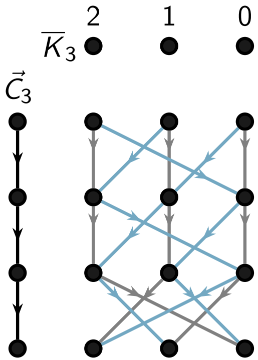

$$F_0 = (id, id, (0,1,2))$$
$$F_1 = ((0,1,2),(0,1,2),(0,2,1))$$

The directed graph $\vec{C}_n \wr \overline{K}_m$

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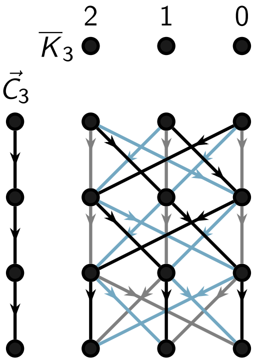


$$F_0 = (id, id, (0,1,2))$$
$$F_1 = ((0,1,2),(0,1,2),(0,2,1))$$
$$F_2 = ((0,2,1),(0,2,1), id)$$

Hamiltonian decomposition of $\vec{C}_n \wr \overline{K}_m$

The existence of a hamiltonian decomposition of $\vec{C}_n \wr \overline{K}_m$ is **equivalent** to the existence of a set of m n-tuples of permutations from S_m that satisfies a very specific set of conditions:

$$\mathcal{F} = \left\{ \begin{array}{cccc} (\mu_{(0,0)}, & \mu_{(0,1)}, & \dots, & \mu_{(0,n-1)}); \\ (\mu_{(1,0)}, & \mu_{(1,1)}, & \dots, & \mu_{(1,n-1)}); \\ & \vdots & & \\ (\mu_{(m-1,0)}, & \mu_{(m-1,1)}, & \dots, & \mu_{(m-1,n-1)}). \end{array} \right\}$$

Back to $\vec{C}_n \wr \vec{C}_3$

Proposition

If n is even, then $\vec{C}_n \wr \vec{C}_3$ is not hamiltonian.

Proof:

1. We show that the existence of a hamiltonian decomposition of $\vec{C}_n \wr \vec{C}_3$ is **equivalent** to the existence of a set of four n-tuples of elements of $(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes S_3$ that must satisfy a very specific set of conditions.

2. Then, we show that no such set of four n-tuples of elements of $(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes S_3$ exists.

The digraph $\vec{C}_n \wr \vec{C}_m$

Theorem

Let $n \geqslant 4$ be an even integer and let $m \geqslant 4$. The digraph $\vec{C}_n \wr \vec{C}_m$ is hamiltonian decomposable.

We prove the statement by constructing the desired decomposition.

The construction

Proof:

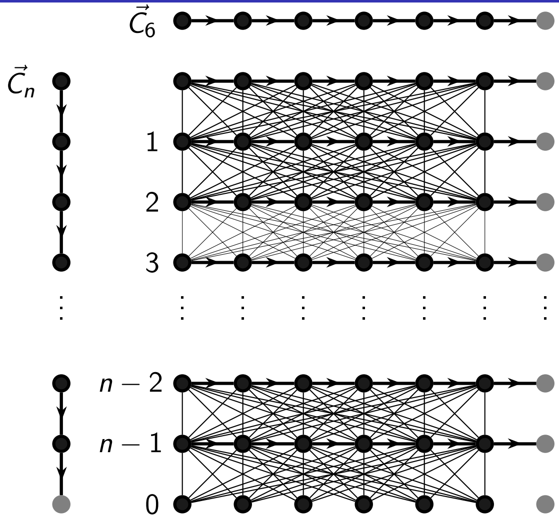
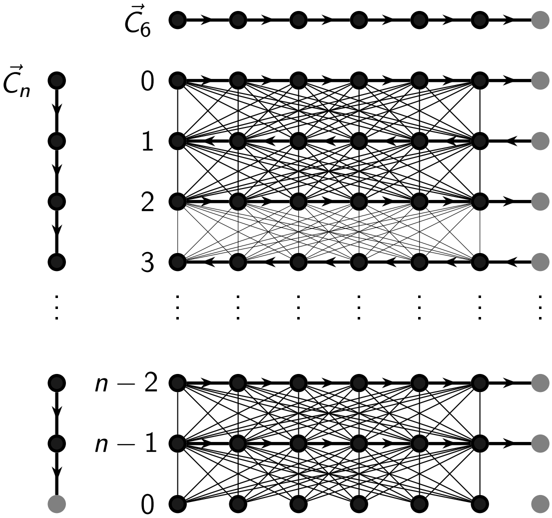


Figure: The wreath product $\vec{C}_n \wr \vec{C}_6$.

The construction

Proof:



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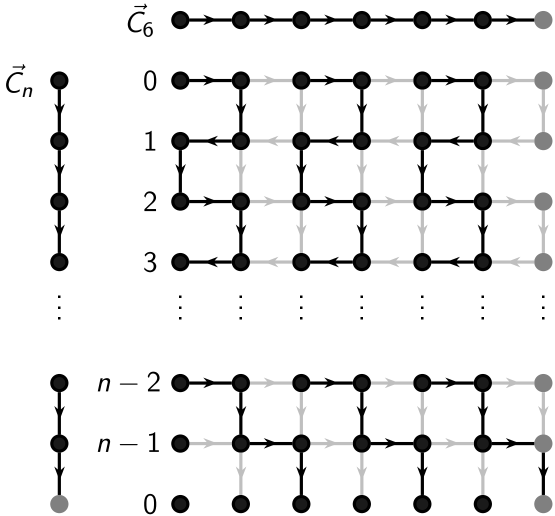

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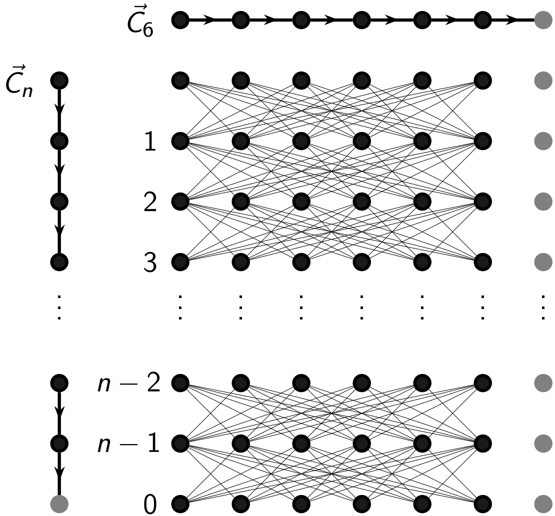

The construction

Proof:

If $n \geqslant 4$, then $\vec{C}_n \times K_m^*$ is hamiltonian decomposable (Paulraja and Sivasankar, 2009).

$\Rightarrow \vec{C}_n \wr \vec{C}_m$ is hamiltonian decomposable. $\square$

Decomposition of $\vec{C}_n \wr K_m^*$

Proposition (L-M (2025))

Let n be even. The digraph $\vec{C}_n \wr K_m^*$ is hamiltonian decomposable if and only if $(n, m) \neq (2, 3)$.

Decomposition of $\vec{C}_n \wr K_m^*$

Proposition (L-M (2025))

Let n be even. The digraph $\vec{C}_n \wr K_m^*$ is hamiltonian decomposable if and only if $(n, m) \neq (2, 3)$.

The following result is a key tool in our construction.

Proposition (Tilson (1980))

Let $m \geqslant 5$. The digraph K_m^* admits a decomposition into directed hamiltonian ***paths****.*

The construction

Proof:

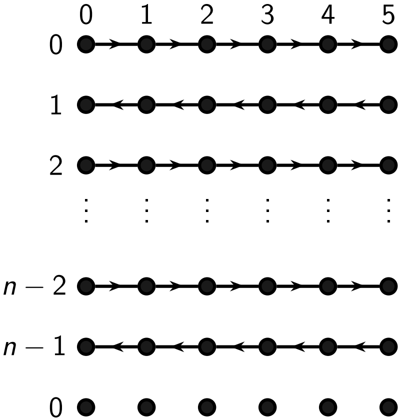

Figure: We start with a hamiltonian dipath of K_m^*.

The construction

Proof:

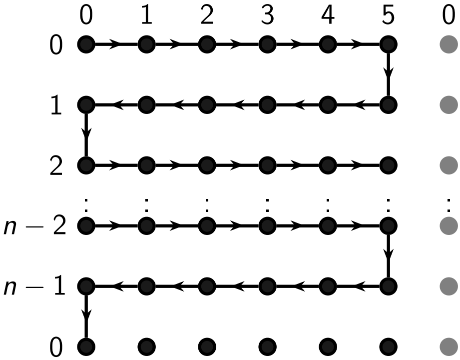

Figure: We then construct a hamiltonian cycle of $\vec{C}_n \wr K_6^*$.

The construction

Proof:

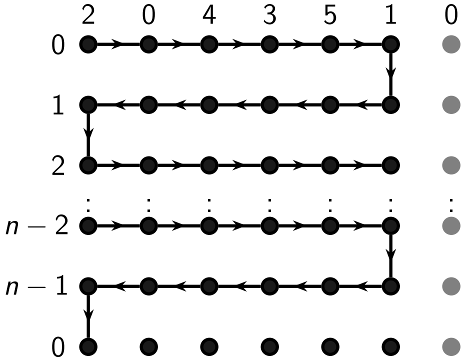

Figure: We construct a second directed hamiltonian cycle of $\vec{C}_n \wr K_6^*$.

The construction

Proof:

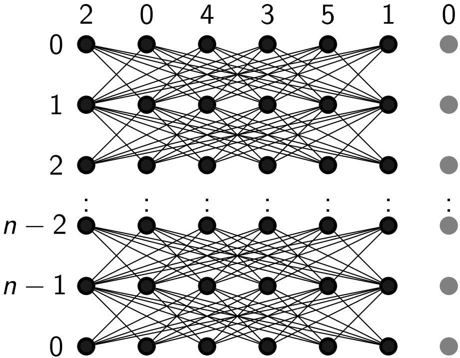

Figure: The categorical product $\vec{C}_n \wr K_6^*$.

Open problems

- Determine when $\vec{C}_n \boxtimes \vec{C}_m$ is hamiltonian decomposable;
- Show that $\vec{C}_2 \times K_m^*$ is hamiltonian decomposable;
- Show that $\vec{C}_n \wr H$ is hamiltonian decomposable when n is even and H is an even ordered hamiltonian decomposable digraph that admits a decomposition into an odd number of cycles.

Thank you