Adapting Häggkvist-style constructions to the directed Oberwolfach problem

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October 9th, 2024

NSERC CRSNG

PIMS

The Oberwolfach problem

The setting: Consider a conference with 7 participants. To facilitate networking, the organizing committee decides to host 3 banquets. The banquet hall has 2 round tables that sit 4 and 3 people, respectively.

The problem: The organizing committee needs a set of 3 seating arrangements (one for each banquet) such that each participant is seated beside every other participants exactly once.

Is this possible?

Graph-theoretic approach

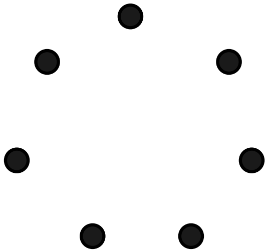

Figure: The empty graph on 7 vertices, $\overline{K}_7$.

Graph-theoretic approach

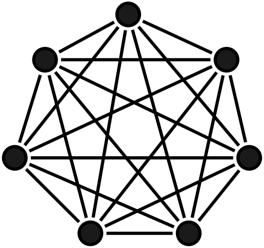

Figure: The complete graph on 7 vertices, K_7.

Graph-theoretic approach

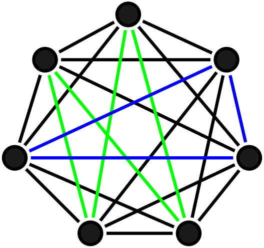

Figure: The first seating arrangement.

Graph-theoretic approach

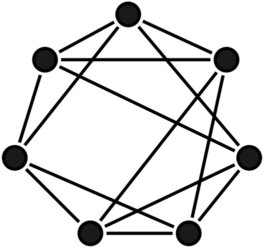

Figure: The complete K_7 minus 1 seating arrangement.

Graph-theoretic approach

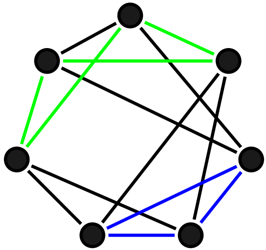

Figure: The second seating arrangement.

Graph-theoretic approach

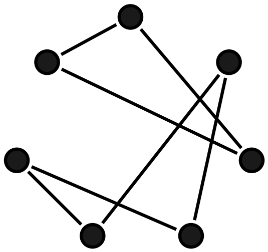

Figure: The third seating arrangement.

The Oberwolfach problem-general case

The setting: Consider a conference with $n = 2k + 1$ participants. The organizing committee decides to host k banquets. The banquet hall has α round tables that sit $m_1, m_2, \ldots, m_\alpha$ participants, respectively, such that $m_1 + m_2 + \cdots + m_\alpha = n$ and each $m_i \geqslant 3$.

The problem: The organizing committee needs a set of k seating arrangements (one for each banquet) such that each participant is seated beside every other participants exactly once.

Is this possible?

Terminology

Definition

A **decomposition** of a graph G is a set of subgraphs $\{H_1, H_2, \ldots, H_k\}$ such that each edge of G appears in exactly one subgraph. We then write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$.

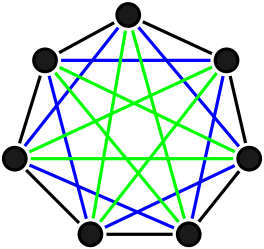

Figure: A decomposition of K_7 in C_7. We see that $K_7 = C_7 \oplus C_7 \oplus C_7$.

Terminology

Definition

A $[m_1, m_2, \ldots, m_\alpha]$-**factor** of G is a spanning subgraph of G comprised to α disjoint cycles of lengths $m_1, m_2, \ldots, m_\alpha$.

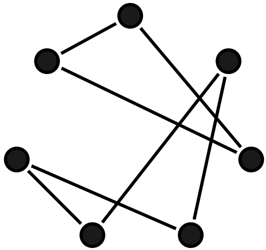

Figure: A $[3, 4]$-factor of K_7.

Terminology

Definition

A $[m_1, m_2, \ldots, m_\alpha]$-**factorization** of G is a decomposition of G into $[m_1, m_2, \ldots, m_\alpha]$-factors.

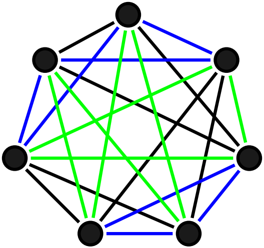

Figure: A $[3, 4]$-factorization of K_7.

The graph-theoretic formulation of the OP

Problem

Let $n = 2k + 1$ and $m_1 + m_2 + \cdots + m_\alpha = n$. Does the graph K_n admit a $[m_1, m_2, \ldots, m_\alpha]$-factorization?

The generalized Oberwolfach problem

What if $n = 2k$?

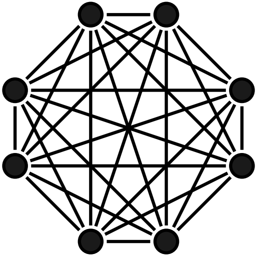

Figure: The complete graph on 8 vertices, K_8.

The generalized Oberwolfach problem

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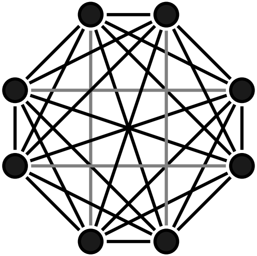

Figure: A 1-factor (perfect matching) of K_8 drawn in grey.

The generalized Oberwolfach problem

What if $n = 2k$?

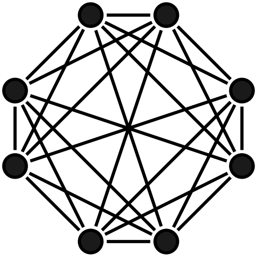

Figure: The graph $K_8 - I$.

The graph-theoretic formulation of the OP

Problem (OP($m_1, m_2, \ldots, m_\alpha$))

Let $m_1 + m_2 + \cdots + m_\alpha = n$. If n is odd, does the graph K_n admit a $[m_1, m_2, \ldots, m_\alpha]$-factorization? If n is even, does the graph $K_n - I$ admit a $[m_1, m_2, \ldots, m_\alpha]$-factorization?

If $m_1 = m_2 = \cdots = m_\alpha = m$, then we write OP($m^\alpha$).

Hamiltonian decomposition of K_n

Theorem (Walecki (1892))

The $OP(n)$ has a solution for all n.

This is a decomposition of K_n or $K_n - I$ into C_n which is also known as a Hamiltonian decomposition.

The Oberwolfach problem with tables of length m

Theorem (Jiaxi (1961), Ray-Chaudhuri and Wilson (1973), Kotzig and Rosa (1974), Baker and Wilson (1977), Brouwer (1978), Rees and Stinson (1987))

The $OP(3^\alpha)$ has a solution if and only if $\alpha \notin \{2, 4\}$

Theorem (Walecki (1892), Alspach and Häggkvist(1985), Alspach et al. (1989), Hoffman and Schellenberg (1991))

If $m \geqslant 4$, then $OP(m^\alpha)$ has a solution.

The Oberwolfach problem with tables of varying lengths

Theorem (Hağgkvist (1985), Bryant and Danziger (2010))

The $OP(m_1, m_2, \ldots, m_\alpha)$ *has a solution when* $m_1, m_2, \ldots, m_\alpha$ *are all even.*

Theorem (Gvozdjak (2004) and Traetta (2013))

The $OP(m_1, m_2)$ *has a solution if and only if* $(m_1, m_2) \notin \{(3,3),(4,5)\}$.

Theorem (Traetta (2024))

The $OP(m_1, m_2, \ldots, m_\alpha)$ *when one of* $m_1, m_2, \ldots, m_\alpha$ *is sufficiently greater than an explicit lower bound.*

The Oberwolfach problem with tables of varying lengths

Theorem (Bryant and Scharaschkin (2009))

The $OP(m_1, m_2, \ldots, m_\alpha)$ *has a solution for infinitely many primes* $n \equiv 1 \pmod{16}$.

Theorem (Alspach et al. (2016))

The $OP(m_1, m_2, \ldots, m_\alpha)$ *has a solution when* $n = 2p$ *where* p *is prime and* $p \equiv 5 \pmod{8}$.

Computational results

Theorem (P. Adams and D. Bryant (2006); A. Deza et al. (2010), F. Franek et al. (2004); F. Franek and A. Rosa.(2006); F. Salassa et al. (2021); M. Meszka (2024))

The $OP(m_1, m_2, \ldots, m_\alpha)$ *has a solution for* $n \leqslant 100$ *except for* $OP(3^2), OP(3^4), OP(4,5)$*, and* $OP(3,3,5)$*.*

Probabilistic approach

Theorem (Glock et al. (2021))

The $OP(m_1, m_2, \ldots, m_\alpha)$ *has a solution for* n *sufficiently large.*

The directed Oberwolfach problem

The setting: Consider a conference with n participants. To facilitate networking, the organizing committee decides to host $n-1$ banquets. The banquet hall has α round tables that sit $m_1, m_2, \ldots, m_\alpha$ participants such that $m_1 + m_2 + \cdots + m_\alpha = n$.

The problem: The organizing committee needs a set of $n-1$ seating arrangements (one for each banquet) such that each participant is seated **to the right** of every other participants exactly once.

Is this possible?

A simple example

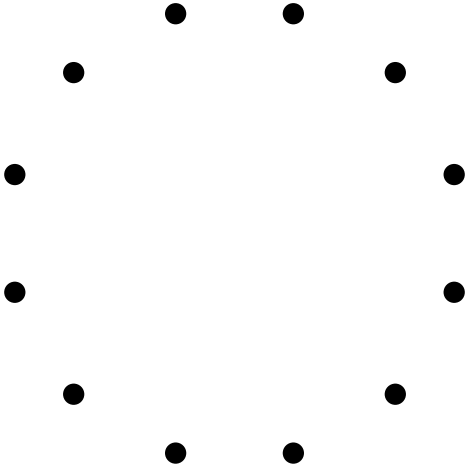

Figure: The 12 participants (one for each vertex).

A simple example

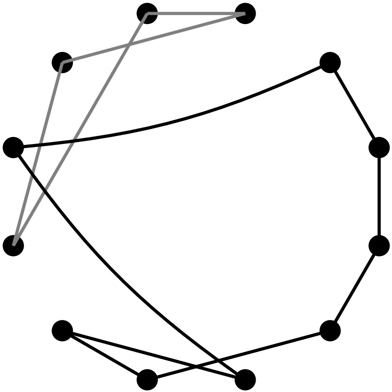

Figure: One seating arrangement with one table of length 4 and one table of length 8.

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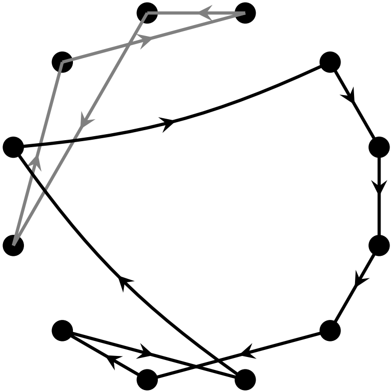

Figure: One seating arrangement with one table of length 4 and one table of length 8.

The complete symmetric digraph

Definition

The **complete symmetric digraph**, denoted K_n^*, is the digraph on n vertices in which for every pair of distinct vertices x and y, there are arcs (x, y) and (y, x).

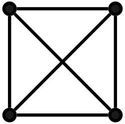

Figure: The complete graph K_4.

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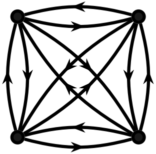

Figure: The complete symmetric digraph K_4^*.

Cycle decomposition

Definition

A $\vec{C}_m$-**factor** of digraph G is a spanning subdigraph of G that is the disjoint union of directed m-cycles.

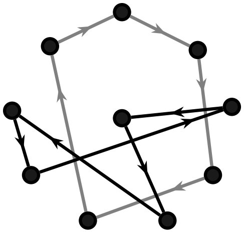

Figure: A $\vec{C}_5$-factor of K^*_{10}.

Cycle decomposition

Definition

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Definition

A $\vec{C}_m$-**factorization** of G is a decomposition of G into $\vec{C}_m$-factors.

$[m_1, m_2, \ldots, m_\alpha]$-factorization

Definition

A **directed** $[m_1, m_2, \ldots, m_\alpha]$-**factor** of digraph G is a spanning subdigraph comprised of disjoint directed cycles of length $m_1, m_2, \ldots, m_\alpha$.

Definition

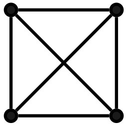
A **directed** $[m_1, m_2, \ldots, m_\alpha]$-**factorization** of digraph G is a decomposition of G into $[m_1, m_2, \ldots, m_\alpha]$-factors.

The graph-theoretic formulation of the directed OP

Problem ($\mathrm{OP}^*(m_1, m_2, \ldots, m_\alpha)$)

Let $m_1, m_2, \ldots, m_\alpha \geqslant 2$. If $m_1 + m_2 + \cdots + m_\alpha = n$, does K_n^* admit a directed $[m_1, m_2, \ldots, m_\alpha]$-factorization?

Why are we not considering two different cases based on the parity of n?



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The graph-theoretic formulation of the directed OP

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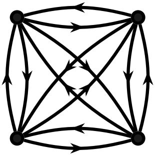

Figure: The complete symmetric digraph K_4^*.

Easy consequences

Corollary (Kadri and Šajna (2024+))

If $(m_1, m_2, \ldots, m_\alpha) \notin \{(4,5), (3,3,5)\}$, *then* $OP^*(m_1, m_2, \ldots, m_\alpha$ *has a solution in each of the following cases:*

- $m_1 = m_2 = \cdots = m_t$;
- $n \leqslant 100$;
- $t = 2$.

We generally consider the case n is even because, when n is odd, a solution to the directed OP can be obtained by orienting a solution to the original OP.

Directed Oberwolfach problem with tables of uniform length

Problem (The directed Oberwolfach problem with tables of uniform length)

To find all integers α and m for which $K^*_{\alpha m}$ admits a $\vec{C}_m$-factorization.

Observe that α =number of cycles in a $\vec{C}_m$-factor.

Previous results (small m or α)

The digraph $K^*_{\alpha m}$ admits a $\vec{C}_m$-factorization when:

- $m = 3$ and $\alpha \neq 2$ (Bermond et al. (1979));
- $\alpha = 1$ and $m \notin \{4, 6\}$ (Tillson (1980));
- $m = 4$ and $\alpha \neq 1$ (Bennett and Zhang (1990); Adams and Bryant, Unpublished);
- $m = 5$ and $\alpha \geqslant 103$ (Abel et al. (2002)).

Previous results (general m)

Theorem (Burgess and Šajna, 2014)

If m is even or α is odd, such that $(\alpha, m) \notin \{(1,6),(1,4)\}$, then $K^*_{\alpha m}$ admits a $\vec{C}_m$-factorization.

We have a solution when tables are of even length or when we have an odd number of tables.

Previous results (general m)

What if we have an even number of tables of odd length?

Theorem (Burgess and Šajna, 2014)

Suppose that α is an even integer and $m \geqslant 3$ is an odd integer. If K^*_{2m} admits a $\vec{C}_m$-factorization, then $K^*_{\alpha m}$ also admits a $\vec{C}_m$-factorization.

It suffices to solve our problem when we have seating arrangements with two tables of odd length.

Previous results (general m)

Conjecture (Burgess and Šajna, 2014)

If m is odd and $m \geqslant 5$, then K^*_{2m} admits a $\vec{C}_m$-factorization.

Theorem (Burgess, Francetić, and Šajna, 2018)

If m is odd and $5 \leqslant m \leqslant 49$, then K^*_{2m} admits a $\vec{C}_m$-factorization.

New Result

Theorem (L-M, 2024)

The digraph K_{2m}^* admits a $\vec{C}_m$-factorization for all odd $m \geqslant 11$.

Tools

Lemma (Burgess and Šajna, 2014)

Let $\{G_1, G_2, \ldots, G_t\}$ be a decomposition of H into spanning subdigraphs. If each G_i admits a directed $[m_1, m_2, \ldots, m_\alpha]$-factorization, then H admits a directed $[m_1, m_2, \ldots, m_\alpha]$-factorization.

Proof Let D_i be the $[m_1, m_2, \ldots, m_\alpha]$-factorization of G_i. We see that

$$\mathcal{F} = \bigcup_{i=1}^{t} D_i$$

is a $[m_1, m_2, \ldots, m_\alpha]$-factorization of H. $\square$

Template

Step 1: Strategically decompose (di)graph G into t spanning sub(di)graphs that fall into r isomorphisms classes: $H_1, H_2, \ldots, H_r$.

Step 2: Show that each isomorphism class admits the desired $[m_1, m_2, \ldots, m_\alpha]$-factorization.

Häggkvist style constructions

Theorem (Hağgkvist (1985))

The $OP(m_1, m_2, \ldots, m_\alpha)$ *has a solution when* $m_1, m_2, \ldots, m_\alpha$ *are all even and* $n \equiv 2 \pmod 4$.

Häggkvist style constructions

Lemma (Häggkvist (1985))

If m is odd, $K_{2m} - I$ admits a decomposition into $\frac{m-1}{2}$ copies of $C_m \wr \overline{K}_2$.

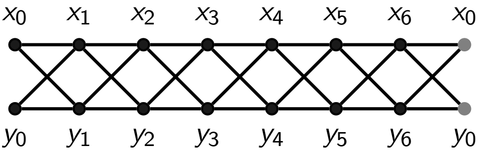

Figure: The graph $C_7 \wr \overline{K}_2$.

Häggkvist style constructions

Lemma (Häggkvist (1985))

If m is odd, $K_{2m} - I$ admits a decomposition into $\frac{m-1}{2}$ copies of $C_m \wr \overline{K}_2$.

Proof: We know that K_m admits a decomposition into $\frac{m-1}{2}$ copies of C_m when m is odd.

We also know that $K_{2m} - I = K_m \wr \overline{K}_2$.

$$\begin{aligned} K_m \wr \overline{K}_2 = & \ (C_m \oplus C_m \oplus \cdots \oplus C_m) \wr \overline{K}_2 \\ = & \ C_m \wr \overline{K}_2 \oplus C_m \wr \overline{K}_2 \oplus \cdots \oplus C_m \wr \overline{K}_2. \end{aligned}$$

□

Lemma (Häggkvist Lemma (1985))

Let $m_1, m_2, \ldots, m_\alpha$ be even integers greater than 2 such that $m_1 + m_2 + \cdots + m_\alpha = 2m$. The graph $C_m \wr \overline{K}_2$ admits a $[m_1, m_2, \ldots, m_\alpha]$-factorization for all $m \geqslant 2$.

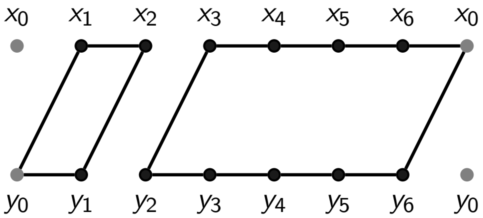

Figure: A $[4, 10]$-factor of $C_m \wr \overline{K}_2$.

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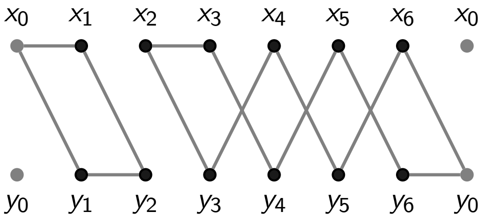

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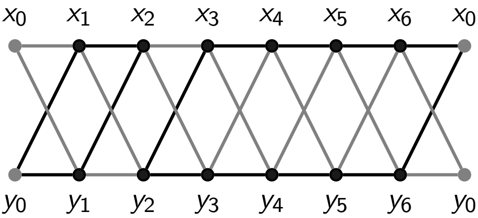

Figure: A $[4, 10]$-factorization of $C_m \wr \overline{K}_2$.

Strategy

Step 1: Decompose K^*_{2m} into $\frac{m-1}{2}$ spanning subdigraphs that fall into one of three isomorphisms classes: G_1, G_2, and G_3.

Step 2: Show that G_1, G_2, and G_3 admit a $\vec{C}_m$-factorization.

Decomposition of K_{2m}^*

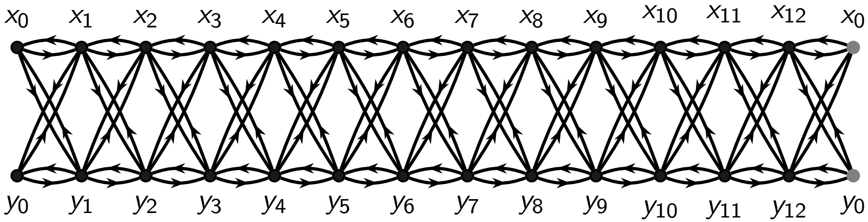

Figure: The spanning digraph $G_1 = \vec{X}(\{\pm 1\}, m) \wr \overline{K}_2$.

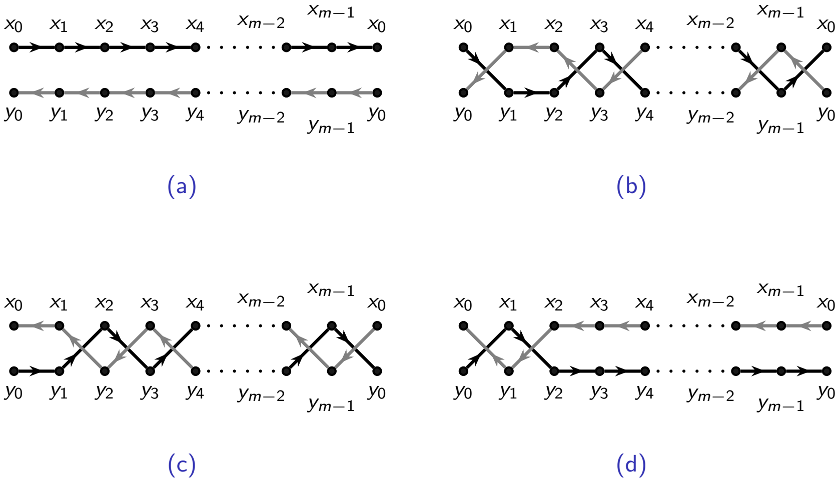

Figure: A $\vec{C}_m$-factorization of G_1.

Result

Proposition

Let $m \geqslant 3$ be an odd integer. The digraph $\vec{X}(\{\pm 1\}, m) \wr \overline{K}_2$ admits a $\vec{C}_m$-factorization.

Decomposition of K_{2m}^*

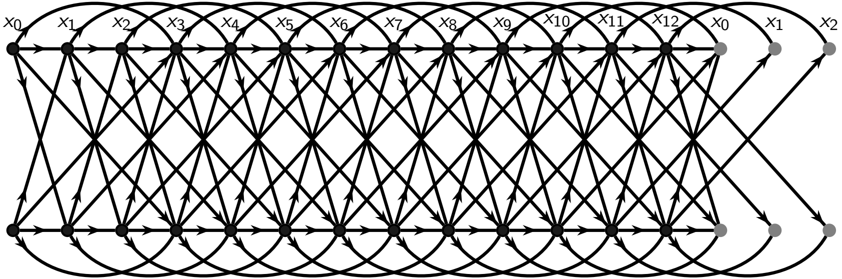

Figure: The spanning digraph $G_2 = \vec{X}(\{1,3\},13) \wr \overline{K}_2$ of $K_{2(13)}^*$.

Key ingredients

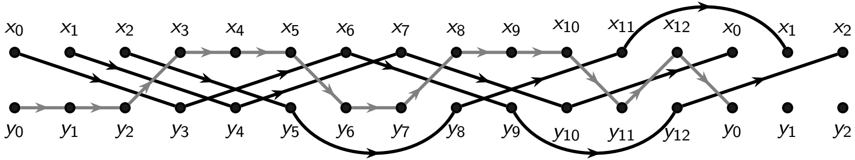

Figure: A $\vec{C}_{13}$-factorization of $\vec{X}(\{1,3\}, m)$ when $m = 13$.

Key ingredients

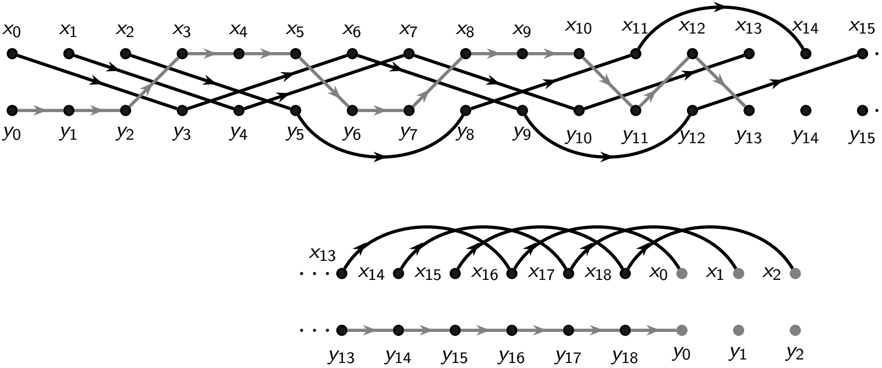

Figure: A $\vec{C}_{19}$-factor of $\vec{X}(\{1,3\},19) \wr \overline{K}_2$.

Key ingredients

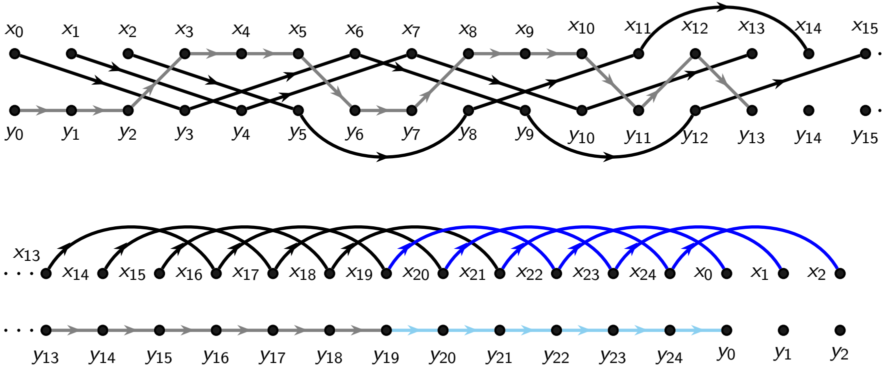

Figure: A $\vec{C}_{25}$-factor of $\vec{X}(\{1,3\},25) \wr \overline{K}_2$.

Result

Proposition

Let $m \geqslant 3$ be an odd integer. The digraph $\vec{X}(\{\pm 1\}, m) \wr \overline{K}_2$ admits a $\vec{C}_m$-factorization.

Proposition

Let $m \geqslant 11$ be an odd integer. The digraph $\vec{X}(\{1, 3\}, m) \wr \overline{K}_2$ admits a $\vec{C}_m$-factorization if and only if $m \not\equiv 3 \pmod 6$.

Decomposition of K_{2m}^*

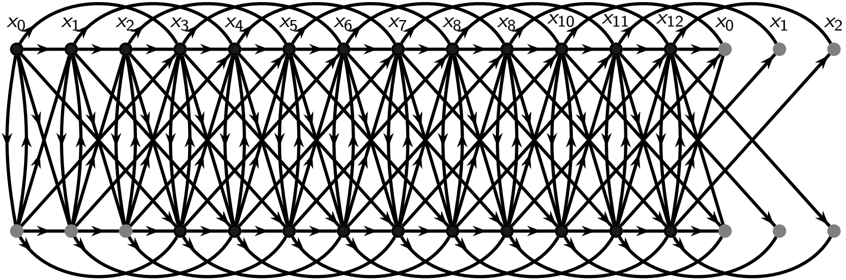

Figure: The spanning digraph $G_3 = \vec{X}(\{1,3\},13) \wr K_2^*$ of $K_{2(13)}^*$.

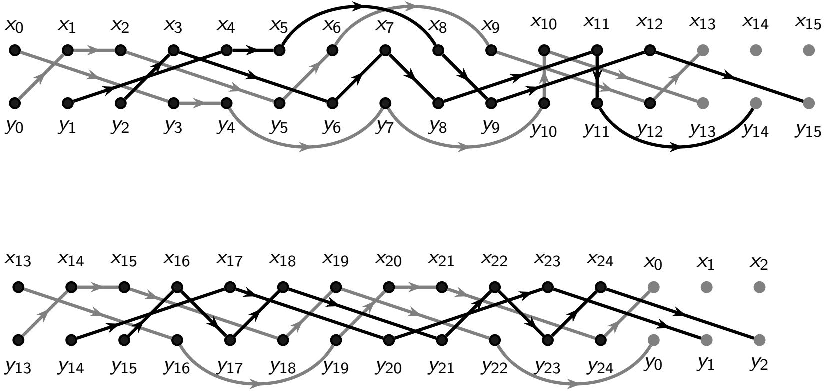

Figure: A $\vec{C}_{25}$-factor of G_3 when $m = 25$.

Result

Proposition

Let $m \geqslant 3$ be an odd integer. The digraph $\vec{X}(\{\pm 1\}, m) \wr \overline{K}_2$ admits a $\vec{C}_m$-factorization.

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Proposition

Let $m \geqslant 11$ be an odd integer such that $m \equiv 1, 5 \pmod 6$. The digraph $\vec{X}(\{1, 3\}, m) \wr K_2^*$ admits a $\vec{C}_m$-factorization.

Summary

Proposition

The digraph K_{2m}^* admits a decomposition into

1. $\frac{m-5}{2}$ *copies of* $\vec{X}(\{\pm 1\}, m) \wr \overline{K}_2$;
2. *one copy of* $\vec{X}(\{1, 3\}, m) \wr \overline{K}_2$;
3. *one copy of* $\vec{X}(\{1, 3\}, m) \wr K_2^*$.

Theorem (L-M, (2024))

If $m \equiv 1, 5 \pmod 6$ *and* $m \geqslant 11$ *then* K_{2m}^* *admits a* $\vec{C}_m$*-factorization.*

Reduction step

Proposition

If K^*_{2m} admits a $\vec{C}_m$-factorization, then $K^*_{2(3^t m)}$ admits a $\vec{C}_{3^t m}$-factorization where t is a positive integer.

If $m' \equiv 3 \pmod 6$ then:

- $m' = 3^t \cdot m$ where $m \equiv 1, 5 \pmod 6$.

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When $m \equiv 1, 5 \pmod 6$ and $m \geqslant 5$, we obtain a $\vec{C}_{m'}$-factorization of $K^*_{2m'}$ using a $\vec{C}_m$-factorization of K^*_{2m}.

When $m = 1$, we use a $\vec{C}_9$-factorization of K^*_{18}.

Main result

Theorem (L-M, (2024))

The digraph K_{2m}^* admits a $\vec{C}_m$-factorization for all odd $m \geqslant 11$.

A complete solution

Theorem

The digraph $K^*_{\alpha m}$ admits $\vec{C}_m$-factorization if and only if $(\alpha, m) \notin \{(1,6),(2,3),(1,4)\}$.

The theorem above is a result of the work of: Bermond and Faber (1976); Bermond, Germa, and Sotteau (1979); Tillson (1980); Bennett and Zhang (1990); Adams and Bryant (Unpublished); Abel, Bennett, and Ge (2002); Burgess and Šajna (2014); Burgess, Francetić, and Šajna (2018); L-M (2024).

The directed Oberwolfach problem with tables of varying lengths

Using a recursive approach, Kadri and Šajna (2024+) obtain several infinite families of solution to $\mathrm{OP}^*(m_1, m_2, \ldots, m_\alpha)$.

Furthermore, they establish the existence of solutions for $n \leqslant 14$ except for three already known exceptions.

Theorem (Kadri and Šajna (2024+)

The $OP^*(m_1, m_2, \ldots, m_\alpha)$ has a solution for $n \leqslant 14$ except for $OP^*(4^1)$, $OP^*(6^1)$, $OP^*(3^2)$.

A key corollary

Theorem (Kadri and Šajna (2024+))

Let $m_1 < m_2$. The OP$^*(m_1, m_2)$ *has a solution except possibly when* $m_1 \in \{4, 6\}$ *and* m_2 *is even.*

Idea: Take a solution to OP$^*(m_1^1)$ and construct a solution to OP$^*(m_1, m_2)$.
Problem: OP$^*(4^1)$ and OP$^*(6^1)$ do not have a solution (Bermond and Faber (1976)).

Result on two tables

Theorem (Horsley and L-M (2024+))

Let $m_1 < m_2$. The $OP^*(m_1, m_2)$ has a solution when $m_1 \in \{4, 6\}$ and m_2 is even.

We construct an $[m_1, m_2]$-factorization of K_n^* when $m_1 + m_2 = n$, $m_1 \in \{4, 6\}$, and m_2 is even.

Approach

Step 1: Decompose K_{2m}^* into $\frac{m-1}{2}$ spanning subdigraphs that fall into one two isomorphisms classes: G_1 and G_2.

Step 2: Show that G_1 and G_2 both admit a $[m_1, m_2]$-factorization.

The first class of digraphs

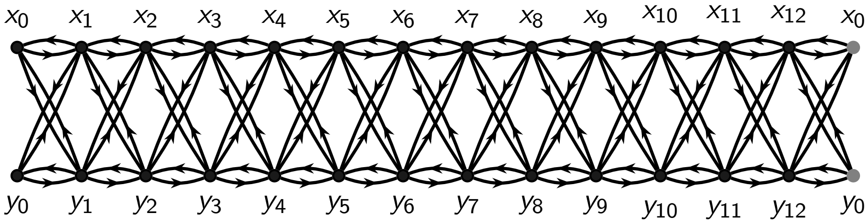

Figure: The spanning digraph $G_1 = \vec{X}(\{\pm 1\}, m) \wr \overline{K}_2$.

The second class of digraph

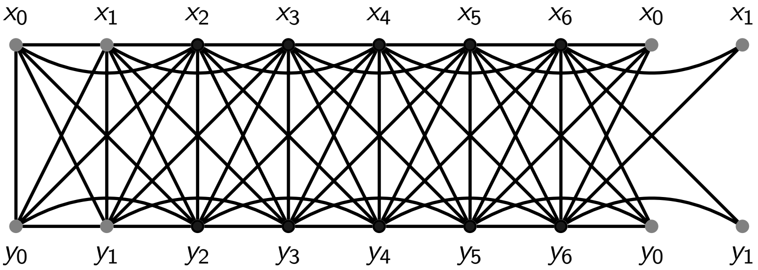

Figure: The digraph $G_2 = \vec{X}(\{\pm 1, \pm 2\}, m) \wr K_2^*$.

Each edge represents a pair of arcs, one for each direction.

A complete solution

Theorem (Kadri and Šajna (2024+) and Horsley and L-M (2024+))

Let $m_1 < m_2$. The $OP^*(m_1, m_2)$ has a solution.

We have a complete solution to the directed Oberwolfach problem with two tables.

Next step

Theorem (Burgess, Danziger, L-M (2024+)))

Let $m_1, m_2, \ldots, m_\alpha$ be even positive integers and $n \equiv 2 \pmod{4}$. The $OP^*(m_1, m_2, \ldots, m_\alpha)$ has a solution.

Next step: To generalize our methods to obtain a solution to $\mathrm{OP}^*(m_1, m_2, \ldots, m_\alpha)$ for any even integers $m_1, m_2, \ldots, m_\alpha$ and $n \equiv 0 \pmod{4}$.

Thanks!