Resolvable directed cycle decompositions of the complete symmetric digraph

Alice Lacaze-Masmonteil
University of Ottawa

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Acknowledgements

I would like to thank:

1. My supervisor Mateja Šajna;
2. University of Ottawa;
3. NSERC.

The complete symmetric digraph

Definition

The **complete symmetric digraph**, *denoted* K_n^*, *is the digraph on n vertices in which for every pair of distinct vertices* x *and* y, *there are arcs* (x, y) *and* (y, x).

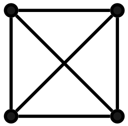

Figure: The complete graph K_4.

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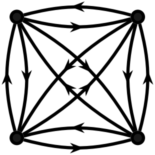

Figure: The complete symmetric digraph K_4^*.

Wreath product

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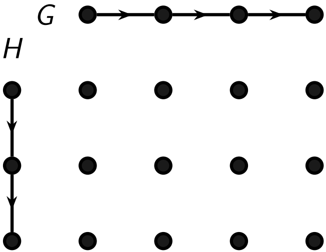
The **wreath product** *of G and H, denoted $G \wr H$, is a digraph on vertex set $V(G) \times V(H)$, where $((x, y), (u, v)) \in A(G \wr H)$ if and only if...*



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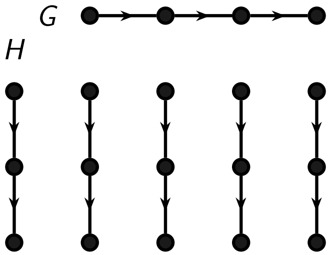

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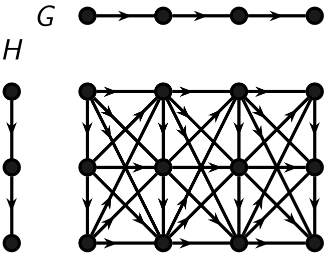

Cycle decomposition

Definition

*A $\vec{C}_m$-**factor** of digraph G is a spanning subdigraph of G that is the disjoint union of directed m-cycles.*

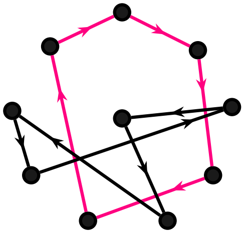

Figure: A $\vec{C}_5$-factor of K^*_{10}.

Definition

A **decomposition** *of G is a set $\{H_1, H_2, \ldots, H_k\}$ of pairwise arc-disjoint subdigraphs of G such that $A(G) = A(H_1) \cup A(H_2) \cup \ldots \cup A(H_k)$.*

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Definition

A $\vec{C}_m$**-factorization** *(or* **resolvable** $\vec{C}_m$**-decomposition***) of* G*, denoted* $R\vec{C}_m$*-D, is a decomposition of* G *into* $\vec{C}_m$*-factors.*

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Problem

To find all integers α and m for which $K_{\alpha m}^*$ admits an R$\vec{C}_m$-D.

Theorem (Bermond, Germa, and Sotteau, 1979)

The digraph $K^*_{3\alpha}$ admits an $R\vec{C}_3$-D if and only if $\alpha \neq 2$.

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Theorem (Bennett and Zhan, 1990; Adams and Bryant, Unpublished)

The directed graph $K_{4\alpha}^*$ admits an $R\vec{C}_4$-D if and only if $\alpha \neq 1$.

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The digraph $K^*_{4\alpha}$ admits an $R\vec{C}_4$-D if and only if $\alpha \neq 1$.

Theorem (Burgess and Šajna, 2014)

If m is even or α is odd, then $K^*_{\alpha m}$ admits an $R\vec{C}_m$-D.

Theorem (Burgess and Šajna, 2014)

Suppose that α is an even integer and m is an odd integer. If K^*_{2m} admits an $R\vec{C}_m$-D, then $K^*_{\alpha m}$ also admits an $R\vec{C}_m$-D.

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Conjecture (Burgess and Šajna, 2014)

If m is odd and $m \geqslant 5$, then K^*_{2m} admits an $R\vec{C}_m$-D.

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If m is odd and $m \geqslant 5$, then K^*_{2m} admits an $R\vec{C}_m$-D.

Theorem (Burgess, Francetić, and Šajna, 2018)

If m is odd and $5 \leqslant m \leqslant 49$, then K^*_{2m} admits an $R\vec{C}_m$-D.

Result

Theorem (Lacaze-Masmonteil and Šajna, 2022+)

Let p be an odd prime such that $p \equiv 5 \pmod{6}$ or $p < 50$. If m is an odd multiple of p and $m \neq 3$, then K_{2m}^* admits an $R\vec{C}_m$-D.

Outline:

• We take a reduction step that narrows down this problem to $m \equiv 1, 5 \pmod{6}$ and m is prime.

Result

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Let p be an odd prime such that $p \equiv 5 \pmod 6$ or $p < 50$. If m is an odd multiple of p and $m \neq 3$, then K^*_{2m} admits an R$\vec{C}_m$-D.

Outline:

- We take a reduction step that narrows down this problem to $m \equiv 1, 5 \pmod 6$ and m is prime.
- For all $m \equiv 5 \pmod 6$, we construct an R$\vec{C}_m$-D of K^*_{2m}.

Proposition

Let t and s be odd integers such that $t, s \geqslant 3$, t is prime and $m = st$. If the graph K_{2t}^* admits an $R\vec{C}_t$-D, then K_{2m}^* admits an $R\vec{C}_m$-D.

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Proof: First, we decompose K^*_{2m} into the following digraphs:

$$K^*_{2m} = K^*_{2t} \wr K^*_s;$$

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Proof: First, we decompose K^*_{2m} into the following digraphs:

$$\begin{aligned} K^*_{2m} &= K^*_{2t} \wr K^*_s; \\ &= (\underbrace{2\vec{C}_t \oplus 2\vec{C}_t \oplus \ldots \oplus 2\vec{C}_t}_{2t-1}) \wr K^*_s; \end{aligned}$$

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Proof: First, we decompose K^*_{2st} into the following digraphs:

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Example: $V(K_{10s}^*) = V(K_s^*)\dot{\cup}V(K_s^*)\dot{\cup}\dots\dot{\cup}V(K_s^*)$.

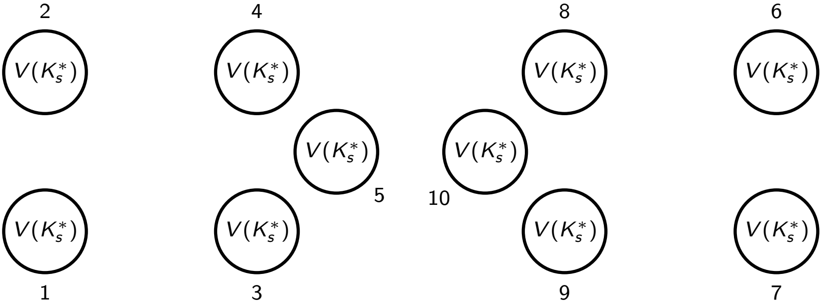

Figure: Partition of $V(K_{10s}^*)$ into 10 sets of $V(K_s^*)$.

$K^*_{10s} = (2\vec{C}_5)\ldots$

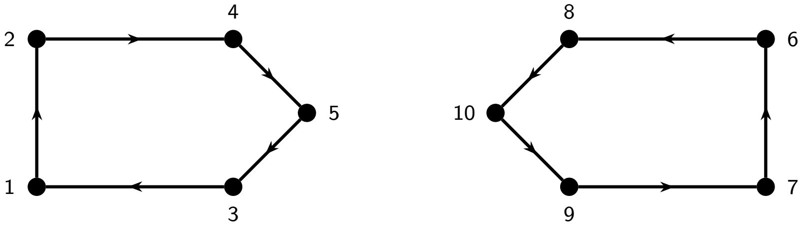

Figure: A $\vec{C}_5$-factor of K^*_{10}.

$$K_{10s}^{*} = (2\vec{C}_5) \wr K_s^{*} \oplus \ldots$$

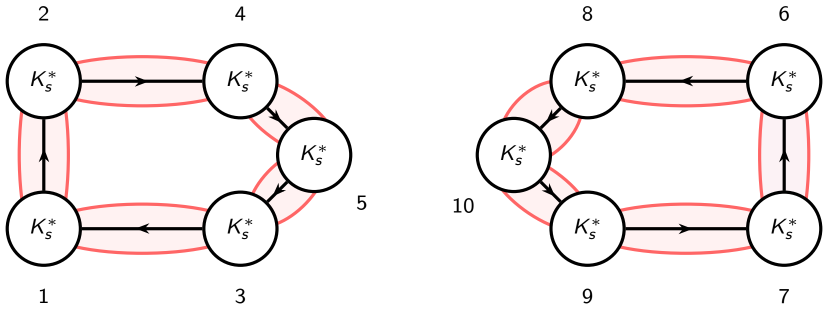

Figure: The subdigraph $(2\vec{C}_5) \wr K_s^{*}$ of K_{10s}^{*}.

$$K_{10s}^* = (2\vec{C}_5) \wr K_s^* \oplus (2\vec{C}_5) \wr \overline{K}_s \ldots$$

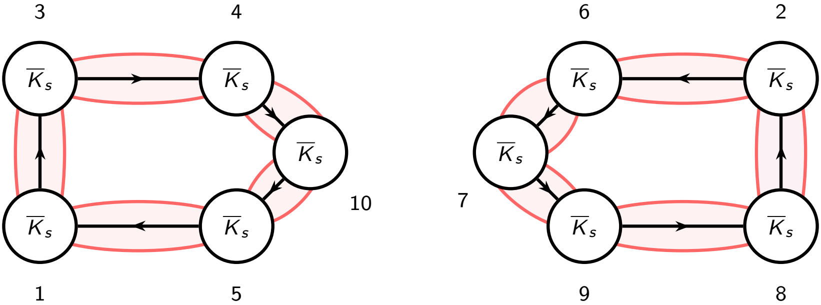

Figure: The subdigraph $(2\vec{C}_5) \wr \overline{K}_s$ of K_{10s}^*.

Proposition

Let t and s be odd integers such that $t, s \geqslant 3$, t is prime and $m = st$. If the graph K^*_{2t} admits an $R\vec{C}_t$-D, then K^*_{2m} admits an $R\vec{C}_m$-D.

Proof:

$$K^*_{2m} = (2\vec{C}_t) \wr K^*_s \oplus \underbrace{(2\vec{C}_t) \wr \overline{K}_s \oplus \ldots \oplus (2\vec{C}_t) \wr \overline{K}_s}_{2t-2}.$$

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The digraph $(2\vec{C}_t) \wr K^*_s$ admits an R$\vec{C}_{st}$-D (Ng, 1998).

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$\Rightarrow$ K_{2m}^* admits an R$\vec{C}_m$-D. □

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Conclusion: To construct an R$\vec{C}_m$-D of K^*_{2m}, it suffices to construct an R$\vec{C}_t$-D of K^*_{2t} for t a prime factor of m.

Theorem (Lacaze-Masmonteil and Šajna, 2022+)

If $m \equiv 5 \pmod 6$, then K_{2m}^* admits an $R\vec{C}_m$-D.

Outline:

- We prove that it suffices to construct a set of 3 $\vec{C}_m$-factors with a particular property.

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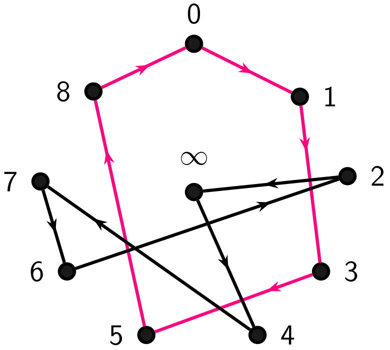
- We prove that it suffices to construct a set of 3 $\vec{C}_m$-factors with a particular property.
- We then construct such a set 3 $\vec{C}_m$-factors.

Labeling the arcs of K_n^*.

1. Let $V(K_n^*) = \mathbb{Z}_{n-1} \cup \{\infty\}$.



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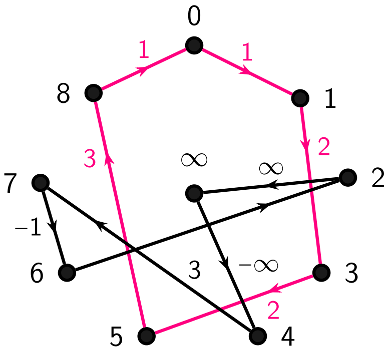
2. Each arc (x, y) is assigned difference base-3, d_i, where $d = y - x \pmod{n-1}$ and ...



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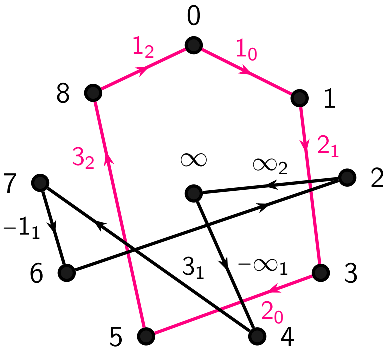

Definition

A set $\mathcal{F}$ of 3 $\vec{C}_m$-factors of K^*_{2m} is 3-**complete** *if each arc in $\mathcal{F}$ has a distinct base-*3 *difference.*

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Proof: Let $V(K_{2m}^*) = \mathbb{Z}_{2m-1} \cup \{\infty\}$. Define the following permutation:

$$\sigma = (\infty)(0, 3, 6, \ldots, 2m{-}4)(1, 4, 7 \ldots, 2m{-}3)(2, 5, 8, \ldots, 2m{-}2).$$

Note that $2m{-}1 \equiv 0 \pmod 3$.

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- Two arcs are in the same orbit if and only if they have the same base-3 difference.

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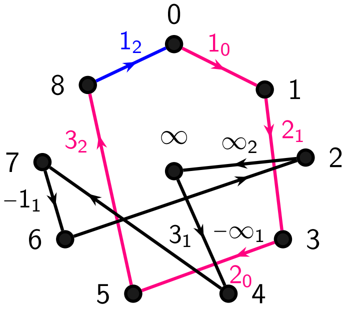

Figure: A $\vec{C}_5$-factor F_0 of K^*_{10}.

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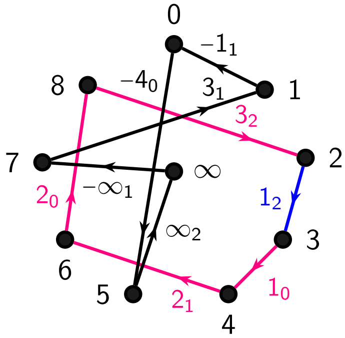

Figure: The $\vec{C}_5$-factor $\sigma(F_0)$.

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$$\sigma = (\infty)(0,3,6,\ldots,2m{-}4)(1,4,7\ldots,2m{-}3)(2,5,8,\ldots,2m{-}2).$$

If $\{F_0, F_1, F_2\}$ is a 3-complete set of $\vec{C}_m$-factors of K^*_{2m}, then:

$$\{\sigma^k(F_i) : k = 0,1,\ldots,\tfrac{2m-4}{3}, i = 0,1,2\}$$

is a $\vec{C}_m$-factorization of K^*_{2m}.

Theorem (Lacaze-Masmonteil and Šajna, 2022+)

Suppose that $m \equiv 5 \ (mod\ 6)$. The digraph K^*_{2m} admits a set of 3-complete $\vec{C}_m$-factors.

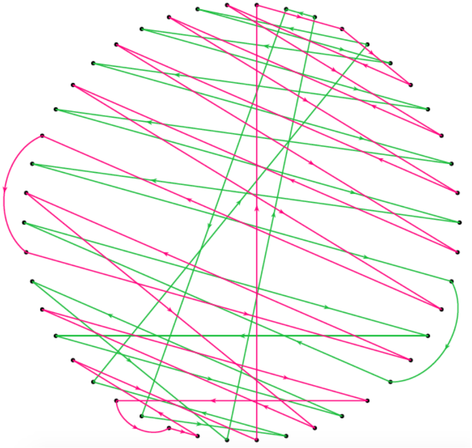

Figure: A $\vec{C}_{23}$-factor of K^*_{46}.

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Theorem (Lacaze-Masmonteil and Šajna, 2022+)

Let α be a positive even integer and p be an odd prime such that $p \equiv 5 \pmod 6$ or $p < 50$. If m is an odd multiple of p, and $m \neq 3$, then $K^*_{\alpha m}$ admits an $R\vec{C}_m$-D.

Thank you!